An example for Green’s Theorem with discontinuous partial derivatives

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Statement of Green’s Theorem

Theorem (G. Green, ≈ 1828)

On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \to \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$.

Then, for any closed rectangle $R$ contained in $\Omega$,

$$\int_{\partial R} \vec{F} = \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$
Proof of Green’s Theorem

Proof.

For \( R = [a, b] \times [c, d] \),

\[
\int_{x=a}^{x=b} P(x, c) \, dx + \int_{y=c}^{y=d} Q(b, y) \, dy - \int_{x=a}^{x=b} P(x, d) \, dx - \int_{y=c}^{y=d} Q(a, y) \, dy = \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} \frac{\partial Q}{\partial x} \, dx \right) \, dy - \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} \frac{\partial P}{\partial y} \, dy \right) \, dx.
\]
Proof of Green’s Theorem

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For \( R = [a, b] \times [c, d] \),

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\int_{x=a}^{x=b} P(x, c) \, dx + \int_{y=c}^{y=d} Q(b, y) \, dy \\
- \int_{x=a}^{x=b} P(x, d) \, dx - \int_{y=c}^{y=d} Q(a, y) \, dy \\
= \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} \frac{\partial Q}{\partial x} \, dx \right) \, dy - \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} \frac{\partial P}{\partial y} \, dy \right) \, dx.
\]
Proof of Green’s Theorem

Proof.

For \( R = [a, b] \times [c, d] \),

\[
\int_{x=a}^{x=b} P(x, c) \, dx + \int_{y=c}^{y=d} Q(b, y) \, dy
\]

\[
- \int_{x=a}^{x=b} P(x, d) \, dx - \int_{y=c}^{y=d} Q(a, y) \, dy
\]

\[
= \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} \frac{\partial Q}{\partial x} \, dx \right) \, dy - \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} \frac{\partial P}{\partial y} \, dy \right) \, dx.
\]

Remark

Nope.
Proof of Green’s Theorem

Proof. 

For \( R = [a, b] \times [c, d] \),

\[
\int_{x=a}^{x=b} P(x, c) \, dx + \int_{y=c}^{y=d} Q(b, y) \, dy \\
- \int_{x=a}^{x=b} P(x, d) \, dx - \int_{y=c}^{y=d} Q(a, y) \, dy \\
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\]

Remark

*Nope. Something's wrong.*
Proof of Green’s Theorem

Proof.

For \( R = [a, b] \times [c, d] \),

\[
\int_{x=a}^{x=b} P(x, c) \, dx + \int_{y=c}^{y=d} Q(b, y) \, dy \\
- \int_{x=a}^{x=b} P(x, d) \, dx - \int_{y=c}^{y=d} Q(a, y) \, dy \\
= \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} \frac{\partial Q}{\partial x} \, dx \right) \, dy - \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} \frac{\partial P}{\partial y} \, dy \right) \, dx.
\]

Remark

\textbf{Nope. Something’s wrong. Let’s go back to the statement of the Theorem.}
Statement of Green’s Theorem

Theorem (G. Green, ≈ 1828)

On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F}: \Omega \to \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$.

Then, for any closed rectangle $R$ contained in $\Omega$,

$$\int_{\partial R} \vec{F} = \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Remark
False!
Statement of Green’s Theorem

Theorem (G. Green, ≈ 1828)

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Remark

False! What’s missing?
Statement of \textbf{Green’s Theorem}

\textbf{Theorem (G. Green, \approx 1828)}

\textit{On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \rightarrow \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$. If $\vec{F} \in C^1(\Omega)$ (meaning that the partial derivatives $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ are continuous on $\Omega$), then, for any closed rectangle $R$ contained in $\Omega$,}

$$\int_{\partial R} \vec{F} = \int_{R} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

\textbf{Remark}
Statement of Green’s Theorem

Theorem (G. Green, ≈ 1828)

On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \to \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$. If $\vec{F} \in C^1(\Omega)$ (meaning that the partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ are continuous on $\Omega$), then, for any closed rectangle $R$ contained in $\Omega$,

$$\int_{\partial R} \vec{F} = \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Remark

This is the usual calculus textbook statement of Green’s Theorem.
Statement of Green’s Theorem


On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \to \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$. If $P$ and $Q$ are continuous on $\Omega$, and the partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ all exist at every point in $\Omega$ except for countably many, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \in L^1_{loc}(\Omega)$, then, for any closed rectangle $R$ contained in $\Omega$,

$$
\int_{\partial R} \vec{F} = \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).
$$
Statement of Green’s Theorem


On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \rightarrow \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$.

If $P$ and $Q$ are continuous on $\Omega$, and the partial derivatives $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ all exist at every point in $\Omega$ except for countably many, and

$\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \in L^1_{loc}(\Omega),$

then, for any closed rectangle $R$ contained in $\Omega$,

$$\int_{\partial R} \vec{F} = \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

**Proof.**

Difficult — comparable to the Looman-Menchoff Theorem in complex analysis.
Statement of Green’s Theorem

**Theorem (Green’s Theorem, in Bruna and Cufí, Complex Analysis)**

On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \rightarrow \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$.

If $P$ and $Q$ are differentiable on $\Omega$, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous on $\Omega$, then, for any closed rectangle $R$ contained in $\Omega$,

$$
\int_{\partial R} \vec{F} = \int_{R} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).
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Statement of Green’s Theorem

Theorem (Green’s Theorem, in Bruna and Cufí, Complex Analysis)

On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \rightarrow \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$.

If $P$ and $Q$ are differentiable on $\Omega$, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous on $\Omega$, then, for any closed rectangle $R$ contained in $\Omega$,

$$\int_{\partial R} \vec{F} = \int_{R} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Proof.

Not as difficult — comparable to the Cauchy-Goursat Theorem in complex analysis.

Differentiable means: can be locally approximated by a linear function.

$\Rightarrow$ all partial derivatives exist, but not conversely.
How about an example?

Is there some $\vec{F} = (P, Q)$ so that $P$ and $Q$ are differentiable, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous, but at least one of the partial derivatives $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ is discontinuous?
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So the Bruna-Cufí version of Green’s Theorem applies, but the usual calculus textbook version does not.
How about an example?

Is there some $\vec{F} = (P, Q)$ so that $P$ and $Q$ are differentiable, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous, but at least one of the partial derivatives $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ is discontinuous?

So the Bruna-Cufí version of Green’s Theorem applies, but the usual calculus textbook version does not.

1-dimensional analogue: $f(x) = x^2 \sin(1/x)$ is differentiable with $f(0) = f'(0) = 0$, but $\lim_{x \to 0} f'(x)$ does not exist: $f'$ is not locally bounded.
Example 1

Let \( \Omega = \) unit disk in \( \mathbb{R}^2 \). \( \vec{F}(x, y) = (P(x, y), Q(x, y)) \). \( \vec{F}(0, 0) = (0, 0) \), so the following example is continuous:
Example 1

Let $\Omega =$ unit disk in $\mathbb{R}^2$. $\vec{F}(x, y) = (P(x, y), Q(x, y))$. $\vec{F}(0, 0) = (0, 0)$, so the following example is continuous:

$$P = y \sqrt{- \ln(x^2 + y^2)}$$

$$Q = x \sqrt{- \ln(x^2 + y^2)}$$

$$\frac{\partial P}{\partial y} = \sqrt{- \ln(x^2 + y^2)} + \frac{y^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{- \ln(x^2 + y^2)}}$$

$$\frac{\partial Q}{\partial x} = \sqrt{- \ln(x^2 + y^2)} + \frac{x^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{- \ln(x^2 + y^2)}}$$
Example 1

Let $\Omega =$ unit disk in $\mathbb{R}^2$. $\vec{F}(x, y) = (P(x, y), Q(x, y))$. $\vec{F}(0, 0) = (0, 0)$, so the following example is continuous:

\[
\begin{align*}
P &= y\sqrt{-\ln(x^2 + y^2)} \\
Q &= x\sqrt{-\ln(x^2 + y^2)} \\
\frac{\partial P}{\partial y} &= \sqrt{-\ln(x^2 + y^2)} + \frac{y^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2 + y^2)}} \\
\frac{\partial Q}{\partial x} &= \sqrt{-\ln(x^2 + y^2)} + \frac{x^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2 + y^2)}} \\
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= \frac{y^2 - x^2}{x^2 + y^2} \cdot \frac{1}{\sqrt{-\ln(x^2 + y^2)}}
\end{align*}
\]
Example 1

Let \( \Omega = \) unit disk in \( \mathbb{R}^2 \). \( \vec{F}(x, y) = (P(x, y), Q(x, y)) \). \( \vec{F}(0, 0) = (0, 0) \), so the following example is continuous:

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    \frac{\partial Q}{\partial x} &= \sqrt{- \ln(x^2 + y^2)} + \frac{x^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{- \ln(x^2 + y^2)}}
\end{align*}
\]

Remark

Close!
Example 1

Let $\Omega =$ unit disk in $\mathbb{R}^2$. $\vec{F}(x, y) = (P(x, y), Q(x, y))$. $\vec{F}(0, 0) = (0, 0)$, so the following example is continuous:

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P = y\sqrt{-\ln(x^2 + y^2)}
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\[
Q = x\sqrt{-\ln(x^2 + y^2)}
\]

\[
\frac{\partial P}{\partial y} = \sqrt{-\ln(x^2 + y^2)} + \frac{y^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2 + y^2)}}
\]

\[
\frac{\partial Q}{\partial x} = \sqrt{-\ln(x^2 + y^2)} + \frac{x^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2 + y^2)}}
\]

**Remark**

Close! Cohen’s version of Green’s Theorem applies. But $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ DNE at $(0, 0)$. 
Example 1

Let $\Omega =$ unit disk in $\mathbb{R}^2$. $\vec{F}(x, y) = (P(x, y), Q(x, y))$. $\vec{F}(0, 0) = (0, 0)$, so the following example is continuous:

$$P = y\sqrt{-\ln(x^2 + y^2)}$$
$$Q = x\sqrt{-\ln(x^2 + y^2)}$$

$$\frac{\partial P}{\partial y} = \sqrt{-\ln(x^2 + y^2)} + \frac{y^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2 + y^2)}}$$

$$\frac{\partial Q}{\partial x} = \sqrt{-\ln(x^2 + y^2)} + \frac{x^2}{x^2 + y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2 + y^2)}}$$

Remark

Close! Cohen’s version of Green’s Theorem applies. But $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ DNE at $(0, 0)$. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ has a removable discontinuity.
Plotting $\vec{F}$ as a vector field:

Figure: `fieldplot([P, Q], x = -.7 .. .7, y = -.7 .. .7);`
Plotting the magnitude $\| \vec{F} \|$ as a scalar:

\[
\text{Figure: } \text{plot3d}(\sqrt{P^2+Q^2}, x = -0.7 .. 0.7, y = -0.7 .. 0.7);
\]
Plotting the partial derivatives as scalars:

\begin{figure}[h]
\centering
\begin{subfigure}{0.49\textwidth}
\centering
\includegraphics[width=\textwidth]{plot1.png}
\end{subfigure}
\begin{subfigure}{0.49\textwidth}
\centering
\includegraphics[width=\textwidth]{plot2.png}
\end{subfigure}
\caption{plot3d(diff(P,y), x = -0.7 .. 0.7, y = -0.7 .. 0.7); plot3d(diff(Q,x), x = -0.7 .. 0.7, y = -0.7 .. 0.7);} 
\end{figure}
Graphics by Maple 18

Plotting $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ as a scalar:

Figure: plot3d(diff(Q,x)-(diff(P,y)),x=-.1 .. .1,y=-.1 .. .1);
Example 2

We want to modify Example 1 to get differentiable $P$ and $Q$, so $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ exist everywhere, but are discontinuous, while $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous.
Example 2

We want to modify Example 1 to get differentiable \( P \) and \( Q \), so \( \frac{\partial P}{\partial y} \) and \( \frac{\partial Q}{\partial x} \) exist everywhere, but are discontinuous, while \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \) is continuous.

**STEP 1:** Introduce a parameter \( 0 < t < \frac{1}{2} \):
\[
\vec{F}_t(x, y) = (P_t(x, y), Q_t(x, y)).
\]
\[
\vec{F}_t(0, 0) = (0, 0),
\]
We want to modify Example 1 to get differentiable $P$ and $Q$, so $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ exist everywhere, but are discontinuous, while $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous.

**STEP 1:** Introduce a parameter $0 < t < \frac{1}{2}$:

\[
\vec{F}_t(x, y) = (P_t(x, y), Q_t(x, y)).
\]
\[
\vec{F}_t(0, 0) = (0, 0),
\]

\[
P_t = y\sqrt{-\ln(x^2 + y^2) \cdot (x^2 + y^2)^t}
\]
\[
Q_t = x\sqrt{-\ln(x^2 + y^2) \cdot (x^2 + y^2)^t}
\]

so Example 1 is the $t \to 0^+$ limit.
Example 2

$P_t$ and $Q_t$ are $C^1$ (⇒ differentiable) with

$$\left[ \frac{\partial P_t}{\partial y} \right]_{(0,0)} = \left[ \frac{\partial Q_t}{\partial x} \right]_{(0,0)} = 0,$$

and:

$$P_t = y \sqrt{-\ln(x^2 + y^2)} \cdot (x^2 + y^2)^t$$

$$Q_t = x \sqrt{-\ln(x^2 + y^2)} \cdot (x^2 + y^2)^t$$

$$\frac{\partial P_t}{\partial y} = \sqrt{-\ln(x^2 + y^2)}(x^2 + y^2)^t - \frac{y^2(1 + 2t \ln(x^2 + y^2))}{(x^2 + y^2)^{1-t} \sqrt{-\ln(x^2 + y^2)}}$$

$$\frac{\partial Q_t}{\partial x} = \sqrt{-\ln(x^2 + y^2)}(x^2 + y^2)^t - \frac{x^2(1 + 2t \ln(x^2 + y^2))}{(x^2 + y^2)^{1-t} \sqrt{-\ln(x^2 + y^2)}}$$
Example 2

$P_t$ and $Q_t$ are $C^1$ ($\xrightarrow{}$ differentiable) with $\frac{\partial P_t}{\partial y}(0,0) = \frac{\partial Q_t}{\partial x}(0,0) = 0$, and:

$$P_t = y\sqrt{-\ln(x^2 + y^2)} \cdot (x^2 + y^2)^t$$

$$Q_t = x\sqrt{-\ln(x^2 + y^2)} \cdot (x^2 + y^2)^t$$

$$\frac{\partial P_t}{\partial y} = \sqrt{-\ln(x^2 + y^2)}(x^2 + y^2)^t - \frac{y^2(1 + 2t \ln(x^2 + y^2))}{(x^2 + y^2)^{1-t} \sqrt{-\ln(x^2 + y^2)}}$$

$$\frac{\partial Q_t}{\partial x} = \sqrt{-\ln(x^2 + y^2)}(x^2 + y^2)^t - \frac{x^2(1 + 2t \ln(x^2 + y^2))}{(x^2 + y^2)^{1-t} \sqrt{-\ln(x^2 + y^2)}}$$

Let $r^2 = x^2 + y^2$. 
Example 2

$P_t$ and $Q_t$ are $C^1$ (⇒ differentiable) with $\frac{\partial P_t}{\partial y}(0,0) = \frac{\partial Q_t}{\partial x}(0,0) = 0$, and:

\[
\begin{align*}
P_t &= y \sqrt{-\ln(x^2 + y^2)} \cdot (x^2 + y^2)^t \\
Q_t &= x \sqrt{-\ln(x^2 + y^2)} \cdot (x^2 + y^2)^t \\
\frac{\partial P_t}{\partial y} &= \sqrt{-\ln(x^2 + y^2)}(x^2 + y^2)^t - \frac{y^2(1 + 2t \ln(x^2 + y^2))}{(x^2 + y^2)^{1-t} \cdot \sqrt{-\ln(x^2 + y^2)}} \\
\frac{\partial Q_t}{\partial x} &= \sqrt{-\ln(x^2 + y^2)}(x^2 + y^2)^t - \frac{x^2(1 + 2t \ln(x^2 + y^2))}{(x^2 + y^2)^{1-t} \sqrt{-\ln(x^2 + y^2)}}
\end{align*}
\]

Let $r^2 = x^2 + y^2$. Max. Value of $f(r) = \sqrt{-\ln(r^2)} r^{2t}$ is $f(e^{-\frac{1}{4t}}) = \frac{1}{\sqrt{2et}}$. 

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STEP 2: Multiply by a smooth “cutoff.” Define $\kappa : (0, \infty) \rightarrow [0, 1]$ to be a weakly decreasing, $C^\infty$ function:

$$\kappa(r) = \begin{cases} 
1 & \text{for } 0 < r < 0.5 \\
0 & \text{for } r > 0.6 
\end{cases}$$
Example 2

**STEP 2:** Multiply by a smooth "cutoff." Define $\kappa : (0, \infty) \rightarrow [0, 1]$ to be a weakly decreasing, $C^\infty$ function:

$$
\kappa(r) = \begin{cases} 
1 & \text{for } 0 < r < 0.5 \\
0 & \text{for } r > 0.6 
\end{cases}
$$

$\vec{F}_t(x, y) = (P_t(x, y), Q_t(x, y))$, now with domain $\mathbb{R}^2$,

$\vec{F}_t(0, 0) = (0, 0),

\begin{align*}
P_t &= y \sqrt{-\ln(x^2 + y^2) \cdot (x^2 + y^2)^t \cdot \kappa(\sqrt{x^2 + y^2})} \\
Q_t &= x \sqrt{-\ln(x^2 + y^2) \cdot (x^2 + y^2)^t \cdot \kappa(\sqrt{x^2 + y^2})}
\end{align*}

Still $C^1$, with large $\frac{\partial P_t}{\partial y}, \frac{\partial Q_t}{\partial x}$ just off-center for small $t$. 
Figure: Plotting the vector field and its magnitude
STEP 3: Pick any sequence of disjoint disks in Quadrant I, with center \((R_k, R_k)\) and radius \(0 < r_k < \frac{R_k}{\sqrt{2}}\), so that \(R_k \to 0^+\) as \(k \to \infty\).

**Figure**: non-overlapping disks approaching the origin in \(\mathbb{R}^2\)
STEP 4: Re-scale $x$, $y$, $z$ directions in the graph of $\vec{F}_t$ by the same factor, $r_k$, by the formula: $r_k \vec{F}_t\left(\frac{x}{r_k}, \frac{y}{r_k}\right)$
STEP 4: Re-scale $x$, $y$, $z$ directions in the graph of $\vec{F}_t$ by the same factor, $r_k$, by the formula: $r_k \vec{F}_t(\frac{x}{r_k}, \frac{y}{r_k})$

**Figure**: shrinking the domain and the height

The same large $\frac{\partial P_t}{\partial y}$, $\frac{\partial Q_t}{\partial x}$ just off-center for small $t$. 
**STEP 5:** Last step! For each $k = 1, 2, 3, \ldots$, re-center a shrunken $\vec{F}_t$ onto disk $\#k$ with center $R_k$, with $t = 2^{-4k} \to 0^+$. 
**STEP 5:** Last step! For each \( k = 1, 2, 3, \ldots \), re-center a shrunken \( \vec{F}_t \) onto disk \( \#k \) with center \( R_k \), with \( t = 2^{-4k} \to 0^+ \).

Also shrink the height again by a factor of \( 2^{-k} \), so that \( \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x} \) have max. value \( 2^{-k} \frac{1}{\sqrt{2e t}} = \frac{2^k}{\sqrt{2e}} \to \infty \):

\[
\vec{F}(x, y) = \sum_{k=1}^{\infty} 2^{-k} r_k \vec{F}_{2^{-4k}} \left( \frac{x - R_k}{r_k}, \frac{y - R_k}{r_k} \right)
\]
**Example 2**

**STEP 5:** Last step! For each $k = 1, 2, 3, \ldots$, re-center a shrunken $\vec{F}_t$ onto disk $\#k$ with center $R_k$, with $t = 2^{-4k} \to 0^+$. Also shrink the height again by a factor of $2^{-k}$, so that $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ have max. value $2^{-k} \frac{1}{\sqrt{2et}} = \frac{2^k}{\sqrt{2e}} \to \infty$:

$$
\vec{F}(x, y) = \sum_{k=1}^{\infty} 2^{-k} r_k \vec{F}_{2^{-4k}} \left( \frac{x - R_k}{r_k}, \frac{y - R_k}{r_k} \right)
$$

**Exercise**

*Still need to check:*

- $\vec{F} = (P, Q)$ is differentiable everywhere, including the origin.
- $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous everywhere, including the origin.
Figure: Easy to check partial derivatives exist at origin because $\mathbf{F} \equiv 0$ along axes!
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