$J$-holomorphic curves in rough almost complex structures

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based on joint work with Yifei Pan and Yuan Zhang

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Holomorphic functions

\[ \mathbb{C} = (\mathbb{R}^2, i) \] has coordinates \( z = x + iy, \bar{z} = x - iy \).

Let \( f \) be a continuous function on a connected open set \( \Omega \subseteq \mathbb{C} \), \( f : \Omega \to \mathbb{C} \), with real/imaginary parts:

\[ f(z) = u(x, y) + iv(x, y) \]

Notation for (classical, pointwise) partial derivatives:

\[ \frac{\partial f}{\partial z} = f_z = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = f_{\bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right). \]
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\]

\( f \) is holomorphic means it satisfies the Cauchy-Riemann Equations:

\[
\begin{bmatrix}
  u_x & u_y \\
  v_x & v_y
\end{bmatrix}
\begin{bmatrix}
  0 & -1 \\
  1 & 0
\end{bmatrix}
= \begin{bmatrix}
  0 & -1 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  u_x & u_y \\
  v_x & v_y
\end{bmatrix}
\implies \begin{bmatrix}
  u_x & u_y \\
  v_x & v_y
\end{bmatrix}
\begin{bmatrix}
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\end{bmatrix}
\begin{bmatrix}
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\end{bmatrix}
\begin{bmatrix}
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  v_x & v_y
\end{bmatrix}
= 0.
\]

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Rough almost complex structures
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- \( \mathcal{C}^\omega \): \( f \) is complex analytic (locally = convergent power series in \( z \))
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- **WUCP**: The Weak Unique Continuation Property: If two solutions $f$ and $g$ satisfy $f \equiv g$ on some open set, then $f \equiv g$
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- **WUCP**: The Weak Unique Continuation Property: If two solutions $f$ and $g$ satisfy $f \equiv g$ on some open set, then $f \equiv g$

- **SUCP**: $f$ has the Strong Unique Continuation Property: If all the derivatives vanish at some point $p$: for all $a, b, c$,

\[
\frac{\partial^a f}{\partial z^a} \bigg|_p = 0 \iff \frac{\partial^{b+c} f}{\partial x^b \partial y^c} \bigg|_p = 0,
\]

then $f \equiv 0$. 
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Generalizations

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1. small changes to the “coefficients” of the differential equations;
2. increase the target dimension to get vector valued $f : \Omega \to \mathbb{C}^n$
To generalize the Cauchy-Riemann Equation: \( \frac{\partial f}{\partial z} \equiv 0 \), by perturbing the coefficients:
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\frac{\partial f}{\partial \bar{z}} = \mu(z) \cdot \frac{\partial f}{\partial z}
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for some complex valued function \( \mu(z) \) with \( \sup |\mu(z)| < 1 \).
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- **Matrix version of generalized C-R equation:**

  \[
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  \end{bmatrix}
  \begin{bmatrix}
  0 & -1 \\
  1 & 0
  \end{bmatrix}
  = J(f(z)) \cdot
  \begin{bmatrix}
  u_x & u_y \\
  v_x & v_y
  \end{bmatrix}
  \]

  for a \( 2 \times 2 \) real matrix \( J \) with:
  - \( J \cdot J = -Id \),
  - entries depending continuously on the coordinates in the target space

  \[
  J(x, y) \approx J_{std} =
  \begin{bmatrix}
  0 & -1 \\
  1 & 0
  \end{bmatrix}
  \]
Generalized analytic functions

Nice properties of solutions of $\frac{\partial f}{\partial \bar{z}} = \mu(z) \cdot \frac{\partial f}{\partial z}$

(under mild hypotheses on $\mu$: measurable, $\|\mu\|_{\infty} < 1$; and on $f$: $W^{1,2}_{loc}$)
Generalized analytic functions

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(under mild hypotheses on \( \mu \): measurable, \( \|\mu\|_\infty < 1 \); and on \( f \): \( W^{1,2}_{loc} \))

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- **SUCP \implies WUCP** (again assuming derivatives exist).
A holomorphic curve is a map $\Omega \rightarrow \mathbb{C}^n$,

$$\vec{f}(z) = [f_1(z), \ldots, f_n(z)] ,$$

where all the components are holomorphic: $\frac{\partial f_k}{\partial z} \equiv 0$
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$$ \begin{bmatrix} df(x, y) \end{bmatrix}_{2n \times 2} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ \vdots \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} df(x, y) \end{bmatrix}_{2n \times 2}. $$
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Holomorphic curves have the same nice properties as holomorphic functions.
Pseudoholomorphic curves — or $J$-holomorphic curves

Modify the $2n \times 2n$ coefficient matrix to get an “Almost Complex Structure” ... $J_{2n \times 2n}$ with real entries depending continuously on the coordinates in $\mathbb{C}^n$, satisfying $J \cdot J = -Id$. 
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A $J$-holomorphic curve $\tilde{f}(z) = [f_1(z), \ldots, f_n(z)]$ is a differentiable map $\Omega \to \mathbb{C}^n = \mathbb{R}^{2n}$ satisfying:

$$d\tilde{f}(x, y) \cdot J_{std} = J(\tilde{f}(x, y)) \cdot d\tilde{f}(x, y).$$
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$$d\vec{f}(x, y) \cdot J_{std} = J(\vec{f}(x, y)) \cdot d\vec{f}(x, y).$$

For $J$ close to $J_{std}$, some linear algebra $\implies$

$$\begin{bmatrix}
\frac{\partial f_1}{\partial \bar{z}} \\
\vdots \\
\frac{\partial f_n}{\partial \bar{z}}
\end{bmatrix} = [Q(f(z))]_{n \times n} \cdot \begin{bmatrix}
\frac{\partial f_1}{\partial z} \\
\vdots \\
\frac{\partial f_n}{\partial z}
\end{bmatrix}.$$ 

for some matrix $Q$ with complex entries, derived from $J$ with the same “regularity”, $Q = 0$ when $J = J_{std}$. 

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Rough almost complex structures
Nice properties of $J$-holomorphic curves

- Local Regularity: If $J$ is $C^{k,\alpha}$, $k = 0, 1, 2, \ldots$, then curves $\vec{f}$ are $C^{k+1,\alpha}$.
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Reference: [Ivashkovich-Shevchishin2011]
Countereamples for uniqueness: \( C^{0,\alpha} \) structure

For \( 0 < \alpha < 1 \), an almost complex structure on \( \mathbb{C}^2 = \mathbb{R}^4 \):

\[
J(z_1, z_2) = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & -2|z_2|^{\alpha} & 0 & -1 \\
-2|z_2|^{\alpha} & 0 & 1 & 0 \\
\end{bmatrix}.
\]
Counterexamples for uniqueness: $\mathcal{C}^{0,\alpha}$ structure

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\end{bmatrix}.$$ 

$C^1$ maps $\mathbb{R}^2 \rightarrow \mathbb{R}^4$:

$$\vec{f}(x, y) = [x, y, 0, 0]$$

$$\vec{g}(x, y) = [x, y, u(x), 0]$$

$\vec{f}$ is $J$-holomorphic, and if $\frac{du}{dx} = 2|u|^\alpha$, then $\vec{g}$ is $J$-holomorphic.
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This ODE doesn’t have unique solutions for initial conditions $u(0) = u'(0) = 0$:

$$u(x) = \begin{cases} 0 & x \leq c \\ (2 - 2\alpha)(x - c))^{1/(1-\alpha)} & x > c \end{cases}$$
Counterexamples for uniqueness: $C^{0,\alpha}$ structure

Conclude: $\vec{f} \equiv \vec{g}$ on an open set but $\vec{f} \not\equiv \vec{g}$,

so for (non-Lipschitz) $J$ with Hölder continuity $C^{0,\alpha}$, the Weak Unique Continuation Property need not hold.
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For $0 < \alpha \leq \frac{2}{3}$, this phenomenon was used by [Ivashkovich-Pinchuk-Rosay2005] to construct an example of an almost complex manifold where $J$ is $\mathcal{C}^{0,\alpha}$ and the Kobayashi-Royden pseudo-norm is not upper semicontinuous.
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Such an example exists for any $0 < \alpha < 1$: [C—Pan2011].
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(still open for Lipschitz or $C^1$ cases)
Counterexamples for uniqueness: $C^0$ structure

**Proposition**

[Rosay2010] *There exist:*

- a complex $2 \times 2$ matrix $Q(z)$ with continuous entries and $Q(0) = [0]$,
- a non-constant, $C^\infty$ smooth map $\vec{g} : \mathbb{C} \rightarrow \mathbb{C}^2$, such that:

$$\frac{\partial \vec{g}}{\partial \bar{z}} = Q(z) \cdot \frac{\partial \vec{g}}{\partial z},$$

and all derivatives of $\vec{g}$ vanish at $z = 0$. ☑
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$\tilde{g}$ can have an isolated zero of $\infty$ order, or a convergent sequence of zeros.
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\frac{\partial \mathbf{\tilde{g}}}{\partial \bar{z}} = \mathbf{Q}(z) \cdot \frac{\partial \mathbf{\tilde{g}}}{\partial z},
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and all derivatives of $\mathbf{\tilde{g}}$ vanish at $z = 0$.

$\mathbf{\tilde{g}}$ can have an isolated zero of $\infty$ order, or a convergent sequence of zeros.

[C—Pan2012]: There exists such a pair $\mathbf{\tilde{g}}, \mathbf{Q}$ where the $\mathbf{Q}$ entries also vanish to infinite order: $z^{-k} \mathbf{Q} \to [0]$ for all $k$. (but $\mathbf{Q}$ is not Lipschitz in any neighborhood of 0)
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This phenomenon can be used to construct a continuous $J$ on $\mathbb{C}^4$ (with $J - J_{std}$ vanishing to infinite order) and $C^\infty$ smooth $J$-holomorphic curves $\vec{f}$ without the SUCP property.
Counterexample for regularity: $\alpha \to 0^+$

**Proposition**

[C—Pan-Zhang2017] There exists a (real) differentiable function $V : \mathbb{C} \to \mathbb{C}$ such that $\frac{\partial V}{\partial \bar{z}}$ is continuous and $\frac{\partial V}{\partial z}$ is discontinuous. □
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Let \( Q(z_1, z_2) = \begin{bmatrix} 0 & \frac{\partial V(z_2)}{\partial \bar{z}_2} \\ 0 & 0 \end{bmatrix} \). Then \( \vec{f}(z) = \begin{bmatrix} f_1(z) \\ z \end{bmatrix} \) \( J \)-holomorphic

\[ \frac{\partial \vec{f}}{\partial z} = Q(\vec{f}(z)) \cdot \frac{\partial \vec{f}}{\partial \bar{z}} \Rightarrow \frac{\partial f_1}{\partial \bar{z}} = \frac{\partial V(z)}{\partial \bar{z}} \Rightarrow f_1(z) = V(z) + C(z) \text{ for some holomorphic } C. \]
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$\implies \frac{\partial \tilde{f}}{\partial z} = Q(\tilde{f}(z)) \cdot \frac{\partial \tilde{f}}{\partial z} \implies \frac{\partial f_1}{\partial z} = \frac{\partial V(z)}{\partial z} \implies f_1(z) = V(z) + C(z)$ for some holomorphic $C$.

So, $J$ is a continuous almost complex structure (but not Hölder), admitting a $J$-holomorphic curve $\tilde{f}$ which is differentiable but not $C^1$. 
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$\Rightarrow \frac{\partial \vec{f}}{\partial z} = Q(\vec{f}(z)) \cdot \frac{\partial \vec{f}}{\partial \bar{z}} \Rightarrow \frac{\partial f_1}{\partial \bar{z}} = \frac{\partial V(z)}{\partial \bar{z}} \Rightarrow f_1(z) = V(z) + C(z)$ for some holomorphic $C$.

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Thank you!

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