1 Real autonomous ODE

The following Lemma gives conditions for the existence of a solution of a differential equation which is bounded on the domain $\mathbb{R}$.

**Lemma 1.1.** Given real numbers $a < b$, if $f : (a, b) \rightarrow \mathbb{R}$ is a continuous, nonvanishing function, and there are some constants $C_1 > 0$, $C_2 > 0$, $\delta_1 \in (0, b - a)$, $\delta_2 \in (0, b - a)$ so that $|f(t)| \leq C_1(t - a)$ for $a < t < a + \delta_1$ and $|f(t)| \leq C_2(b - t)$ for $b - \delta_2 < t < b$, then there exists a one-to-one, onto function $g : \mathbb{R} \rightarrow (a, b)$ so that $y = g(t)$ is a solution of the equation $\frac{dy}{dt} = f(y)$.

**Proof.** $\frac{1}{f(x)}$ is continuous on $(a, b)$, so the function

$$G(t) = \int_{a+b}^{t} \frac{1}{f(x)} dx$$

is differentiable on $(a, b)$ with a nonvanishing, nonzero derivative, $\frac{d}{dt} G(t) = \frac{1}{f(t)}$. Because $f$ and $\frac{1}{f}$ have constant sign, $G(t)$ is monotone on $(a, b)$. Suppose $f(t) > 0$, so $G$ is increasing; the $f(t) < 0$ case is similar.

For $t \in (b - \delta_2, b)$,

$$G(t) = \int_{a+b}^{b-\delta_2} \frac{1}{f(x)} dx + \int_{b-\delta_2}^{t} \frac{1}{f(x)} dx \geq \int_{a+b}^{b-\delta_2} \frac{1}{f(x)} dx + \int_{b-\delta_2}^{t} \frac{1}{C_2(b - x)} dx,$$

*Supported in part by National Science Foundation DMS-1265330.*
which is unbounded. Similarly, $G$ is also unbounded at the other endpoint, so $G : (a, b) \to \mathbb{R}$ is onto and invertible. Let $T$ be any constant, and define

$$g(t) = G^{-1}(t + T),$$

so $g : \mathbb{R} \to (a, b)$ is onto and increasing, and (by the Inverse Function Theorem, [C]), $y = g(t)$ is differentiable with

$$\frac{dy}{dt} = \frac{1}{G'(G^{-1}(t + T))} = \frac{1}{f(y)} = f(y).$$

Such solutions with domain $\mathbb{R}$ are unique up to translation.

**Lemma 1.2.** Given an open (possibly infinite) interval $I$, if $f : I \to \mathbb{R}$ is a continuous, nonvanishing function, and $g_1 : \mathbb{R} \to I$ and $g_2 : \mathbb{R} \to I$ are solutions of the equation $\frac{dy}{dt} = f(y)$, then there exists a constant $T$ so that $g_2(t) = g_1(t + T)$.

**Proof.** Because $g'_1(x) = f(g_1(x))$ is continuous and nonzero, the Inverse Function Theorem applies. For $t \in \mathbb{R}$,

$$\frac{d}{dt} \left( g_1^{-1}(g_2(t)) - t \right) = \frac{1}{g'_1(g_1^{-1}(g_2(t))))} g'_2(t) - 1 = \frac{1}{f(g_1(g_1^{-1}(g_2(t))))} f(g_2(t)) - 1 \equiv 0.$$
There is also a local uniqueness theorem for solutions on an interval, with one initial condition.

**Lemma 1.3.** Given open (possibly infinite) intervals $I_0$, $I_1$, $I_2$, if $f : I_0 \to \mathbb{R}$ is a continuous, nonvanishing function, and $g_1 : I_1 \to I_0$ and $g_2 : I_2 \to I_0$ are solutions of the equation $\frac{du}{dt} = f(y)$, and there is a point $c \in I_1 \cap I_2$ such that $g_1(c) = g_2(c)$, then there exists $\delta > 0$ so that $g_1(t) = g_2(t)$ for all $t \in (c - \delta, c + \delta)$.

**Proof.** Because $g_1'(x) = f(g_1(x))$ is continuous and nonzero, the Inverse Function Theorem applies: there exists some $\delta_1 > 0$ so that $g_1$ is one-to-one on $(c - \delta_1, c + \delta_1)$. Suppose $f > 0$, so $g_1$ is increasing; the $f < 0$ case is similar. Let $\varepsilon = \min\{g_1(c + \frac{1}{2}\delta_1) - g_1(c), g_1(c) - g_1(c - \frac{1}{2}\delta_1)\} > 0$. Because $g_2$ is continuous, there is some $\delta_2 > 0$ corresponding to $\varepsilon$, so that for all $t \in (c - \delta_2, c + \delta_2)$, $|g_2(t) - g_2(c)| = |g_2(t) - g_1(c)| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$, then $(c - \delta, c + \delta) \subseteq I_1 \cap I_2$, where both $g_1$ and $g_2$ are defined. Also, for any $t \in (c - \delta, c + \delta)$,

$$g_1(c - \frac{1}{2}\delta_1) \leq g_1(c) - \varepsilon < g_2(t) < g_1(c) + \varepsilon \leq g_1(c + \frac{1}{2}\delta_1),$$

and by the Intermediate Value Theorem, there is some $x \in (c - \frac{1}{2}\delta_1, c + \frac{1}{2}\delta_1)$ so that $g_1(x) = g_2(t)$; this shows that $x = g_1^{-1}(g_2(t))$, so $g_2(t)$ is in the domain of $g_1^{-1}$. As in (3), for $c - \delta < t < c + \delta$,

$$\frac{d}{dt}(g_1^{-1}(g_2(t))) = 1 \implies g_1^{-1}(g_2(t)) = t + T$$

for some constant $T$. Evaluating $g_2(t) = g_1(t + T)$ at $t = c$ gives $g_2(c) = g_1(c + T)$, and $g_2(c) = g_1(c)$ by hypothesis, so $c + T = c$ because $g_1$ is one-to-one. It follows that $T = 0$ and $g_1(t) = g_2(t)$ for all $t \in (c - \delta, c + \delta)$. \blacksquare
Lemma 1.4. If \( f : \mathbb{R} \to \mathbb{R} \) is a continuous, nonvanishing function, then there exist some open interval \( I \) and a one-to-one, onto function \( g : I \to \mathbb{R} \) so that \( y = g(t) \) is a solution of the equation \( \frac{dy}{dt} = f(y) \).

Proof. \( \frac{1}{f(x)} \) is continuous on \( \mathbb{R} \), so the function

\[
G(t) = \int_0^t \frac{1}{f(x)} \, dx
\]

is differentiable on \( \mathbb{R} \) with a nonvanishing, nonzero derivative, \( \frac{d}{dt} G(t) = \frac{1}{f(t)} \).

Because \( f \) and \( \frac{1}{f} \) have constant sign, \( G(t) \) is monotone on \( \mathbb{R} \). Its image is some open interval \( I \), so \( G : \mathbb{R} \to I \) is invertible.

Let \( T \) be any constant, and define \( g(t) = G^{-1}(t + T) \) on the interval \( I - T = \{ x \in \mathbb{R} : x + T \in I \} \), so \( g : I - T \to \mathbb{R} \) is invertible, and therefore not bounded. \( g \) is a solution of the ODE as in Lemma 1.1.

Lemma 1.5. Given \( b \in \mathbb{R} \), if \( f : (-\infty, b) \to \mathbb{R} \) is a continuous, nonvanishing function, and there are some constants \( C_3 > 0, \delta_3 \in (0,1) \) so that \( |f(t)| \leq C_3(b - t) \) for \( b - \delta_3 < t < b \), then there exist an open interval \( I \) and a one-to-one, onto function \( g : I \to (-\infty, b) \) so that \( y = g(t) \) is a solution of the equation \( \frac{dy}{dt} = f(y) \).

Proof. \( \frac{1}{f(x)} \) is continuous on \( (-\infty, b) \), so the function

\[
G(t) = \int_{b-1}^t \frac{1}{f(x)} \, dx
\]

is differentiable on \( (-\infty, b) \) with a nonvanishing, nonzero derivative, \( \frac{d}{dt} G(t) = \frac{1}{f(t)} \). Because \( f \) and \( \frac{1}{f} \) have constant sign, \( G(t) \) is monotone on \( (-\infty, b) \).

Suppose \( f(t) > 0 \), so \( G \) is increasing; the \( f(t) < 0 \) case is similar.

For \( t \in (b - \delta_3, b) \),

\[
G(t) = \int_{b-1}^{b-\delta_3} \frac{1}{f(x)} \, dx + \int_{b-\delta_3}^t \frac{1}{f(x)} \, dx \geq \int_{b-1}^{b-\delta_3} \frac{1}{f(x)} \, dx + \int_{b-\delta_3}^t \frac{1}{C_3(b - x)} \, dx,
\]

which is unbounded. So, the image of \( G \) is either \( I = (L, \infty) \) or \( I = \mathbb{R} \).

Let \( T \) be any constant, and define \( g(t) = G^{-1}(t + T) \) on the interval \( I - T = \{ x \in \mathbb{R} : x + T \in I \} \), so \( g : I - T \to (-\infty, b) \) is invertible, and therefore not bounded. \( g \) is a solution of the ODE as in Lemma 1.1.
Theorem 1.6. Let $f : \mathbb{R} \to \mathbb{R}$ be a real analytic function. The following are equivalent:

1. there exists a non-constant, bounded, real analytic function $g : \mathbb{R} \to \mathbb{R}$ so that $y = g(t)$ is a solution of the equation $\frac{dy}{dt} = f(y)$;

2. there are at least two distinct points $a_0$, $b_0$ where $f(a_0) = f(b_0) = 0$.

Proof. At any zero of $f$, say $c_0$, the real analyticity implies there is some constant $C_0$ so that $|f(t)| \leq C_0|x - c_0|$ for $x$ near $c_0$.

To show 2. $\implies$ 1., the property that $f$ is real analytic on $\mathbb{R}$ implies that $a_0$ is an isolated zero, and because $a_0$ is not the only zero of $f$, there is some open interval $(a, b)$ with either $a = a_0$ or $b = a_0$, satisfying the hypotheses of Lemma 1.1. The conclusion is that there exists a non-constant solution $g : \mathbb{R} \to (a, b)$, which by construction, (1) and (2), is real analytic.

To show 1. $\implies$ 2., suppose, toward a contradiction, that there exists a solution $g_1$ as claimed, and that $f$ has fewer than two zeros.

Case 1. If $f$ is nonvanishing on $\mathbb{R}$ then Lemma 1.4 applies and there is some nonempty open interval $I$ and some onto function $g_2 : I \to \mathbb{R}$ which is a solution of $\frac{dy}{dt} = f(y)$. By the construction of Lemma 1.4, $g_2$ is real analytic. On the interval $I$, $g_2 - g_1$ is continuous, and because $g_1$ is bounded and $g_2$ is onto, $g_2 - g_1$ attains some negative value and some positive value, so by the Intermediate Value Theorem, there is some $c$ so that $g_1(c) = g_2(c)$. By Lemma 1.3, $g_1(t) \equiv g_2(t)$ on some interval $(c - \delta, c + \delta)$, and because both functions are real analytic, $g_1(t) \equiv g_2(t)$ on $I$. The contradiction is that $g_2$ is unbounded on $I$ while $g_1$ is bounded.

Case 2. If $f$ has exactly one zero, $b \in \mathbb{R}$, then Lemma 1.5 applies to $f$ on $(-\infty, b)$: there is some interval $I_2$ and some one-to-one, onto solution $g_2 : I_2 \to (-\infty, b)$. By an analogous existence result for $f$ on $(b, \infty)$, there is some interval $I_3$ and some one-to-one, onto solution $g_3 : I_3 \to (b, \infty)$. Because $g_1$ is non-constant, there is some $x_0$ where $g(x_0) \neq b$.

If $g_1(x_0) < b$, then there is some $T \in I_2$ with $g_2(T) = g_1(x_0)$, and the function $g_4(t) = g_2(t + T - x_0)$ is, by construction, a real analytic solution of $\frac{dy}{dt} = f(y)$ on an open interval $I_2 - (T - x_0)$ satisfying $g_4(x_0) = g_1(x_0)$. As in Case 1., the uniqueness from Lemma 1.3 shows that $g_4 = g_1$ on $I - (T - x_0)$, contradicting the boundedness of $g_1$.

The $g_1(x_0) > b$ case is similar, using $g_3$. $\blacksquare$
2 Ordinary differential inequalities

2.1 Linear differential inequalities

Lemma 2.1. If $u(t)$ is continuous on $[a, b]$ and $u(a) > 0$, then either $u(t) > 0$ for all $t \in [a, b]$, or there is some $t_1 \in (a, b]$ so that $u(t) > 0$ on $[a, t_1)$ and $u(t_1) = 0$.

Proof. Consider the set $S = \{x \in [a, b] : u(t) > 0$ for all $t \in [a, x]\}$; it is non-empty (by continuity of $u$ and $u(a) > 0$), and bounded above, so it has a least upper bound $t_1 \in (a, b]$. If there were some $t_0$ with $a < t_0 < t_1$ and $u(t_0) \leq 0$, then there would be no $x \in S$ with $x > t_0$, and $t_0$ would be an upper bound, contradicting the least property of $t_1$. So, $u(t) > 0$ for all $t \in [a, t_1)$. Case 1: If $u(t_1) = 0$, then $u > 0$ on $[a, t_1)$ as claimed. Case 2: If $u(t_1) < 0$ then by continuity, there is some $t_2$ with $a < t_2 < t_1$ and $u(t_2) < 0$, contradicting the above property that $u(t) > 0$ for all $t \in [a, t_1)$. Case 3: If $t_1 = b$ and $u(t_1) > 0$, then $u > 0$ on $[a, b]$ as claimed. Case 4: If $t_1 < b$ and $u(t_1) > 0$, then by continuity, $u(t)$ would be positive on some interval $[a, t_1 + \delta)$, contradicting the property that $t_1$ is an upper bound for $S$. Only Cases 1 and 3 do not lead to a contradiction. ■

Lemma 2.2. Suppose $a(t)$ is a real function on $[0, 1]$ such that $a(t) \geq 0$ and $a$ is bounded on every subinterval $[0, x] \subseteq [0, 1]$. If $y$ is continuous on $[0, 1]$ with $y(0) \geq 0$, $\lim_{t \to 0^+} y'(t) \geq 0$, and $y''(t) \geq a(t)y(t)$ for $0 < t < 1$, then $y(t) \geq 0$ for all $0 \leq t \leq 1$ and $y'(t) \geq 0$ for all $0 < t < 1$.

Proof. Case 1: $y(0) > 0$ and $\lim_{t \to 0^+} y'(t) > 0$. In this case, we can show that $y > 0$ on $[0, 1]$ and $y' > 0$ on $(0, 1)$. By Lemma 2.1 applied to $y$ on $[0, 1]$, either $y > 0$ on $[0, 1]$, or there is some $t_1 \in (0, 1]$ so that $y(t) > 0$ for all $t \in [0, t_1)$ and $y(t_1) = 0$. In the latter case, $y$ attains some positive maximum value on $[0, t_1]$. If $y(0)$ is the maximum, then by the Mean Value Theorem, for any $0 < \delta < t_1$, there is some $t_2$ with $0 < t_2 < \delta$ and $y'(t_2) = \frac{y(\delta) - y(0)}{\delta} \leq 0$, which contradicts $y'(t) > 0$. If the maximum is at an interior point $t_3$ with $0 < t_3 < t_1$, then $y'(t_3) = 0$. From $\lim_{t \to 0^+} y'(t) > 0$, there is some $t_4$ with $0 < t_4 < t_3$ and $y'(t_4) > 0$. Applying the Mean Value Theorem to $y'$ on $[t_4, t_3]$, there is some $t_5$ with $t_4 < t_5 < t_3$ and $y''(t_5) = \frac{y'(t_3) - y'(t_4)}{t_3 - t_4} < 0$. This contradicts $y''(t_5) \geq a(t_5)y(t_5) \geq 0$. We can conclude that $y(t_1)$ must be the
maximum value, and \( y(t_1) > 0 \), which contradicts \( y(t_1) = 0 \). This shows \( y(t) > 0 \) for all \( t \in [0, 1] \).

For any \( t_7 \in (0, 1) \), there is some \( t_8 \) with \( 0 < t_8 < t_7 \) and \( y'(t_8) > 0 \). By the Mean Value Theorem, there is some \( t_9 \) with \( \frac{y'(t_7) - y'(t_8)}{t_7 - t_8} = y''(t_9) \geq a(t_9)y(t_9) \geq 0 \). It follows that \( y'(t_7) \geq y'(t_8) > 0 \).

Case 1 did not use the boundedness of \( a \), just \( a \geq 0 \).

Case 2: \( y(0) \geq 0 \) and \( \lim_{t \to 0^+} y'(t) \geq 0 \). Suppose, toward a contradiction, that there is some \( t_0 \) with \( 0 < t_0 < 1 \) and \( y(t_0) < 0 \). For \( 0 \leq t \leq t_0 \), there is a bound \( A > 0 \) with \( 0 \leq a(t) \leq A \). For \( t \in [0, t_0] \), define \( u(t) = y(t) - \frac{1}{2}y(t_0)e^{\sqrt{A(t-t_0)}} \). Then, by construction, \( u(0) > 0 \) and \( u(t_0) < 0 \). For \( 0 < t < t_0 \),

\[
\begin{align*}
u'(t) &= y'(t) - \frac{1}{2}y(t_0)\sqrt{A}e^{\sqrt{A(t-t_0)}} \\
\implies \lim_{t \to 0^+} u'(t) &= \left( \lim_{t \to 0^+} y'(t) \right) - \frac{1}{2}y(t_0)\sqrt{A} > 0, \\
u''(t) &= y''(t) - \frac{1}{2}y(t_0)Ae^{\sqrt{A(t-t_0)}} \\
&\geq \frac{a(t)y(t)}{2} + \frac{1}{2}y(t_0)a(t)e^{\sqrt{A(t-t_0)}} - \frac{1}{2}y(t_0)Ae^{\sqrt{A(t-t_0)}} \\
&= \frac{a(t)y(t)}{2} + \frac{1}{2}y(t_0)(a(t) - A)e^{\sqrt{A(t-t_0)}} \\
&\geq a(t)u(t).
\end{align*}
\]

Let \( w(t) = u(t_0t) \), just horizontally re-scaling \( u \) to the domain \([0, 1]\) so that \( w(0) > 0 \), \( w(1) < 0 \), \( \lim_{t \to 0^+} w'(t) > 0 \), and \( w''(t) \geq t_0^2a(t_0t)w(t) \), so Case 1 applies to \( w \), contradicting \( w(1) < 0 \). The conclusion is that \( y(t) \geq 0 \) on \([0, 1]\), and on \([0, 1]\) by continuity.

To establish the inequality \( y' \geq 0 \), for any \( t_1 \) with \( 0 < t_1 < 1 \) and any \( \epsilon > 0 \), from \( \lim_{t \to 0^+} y'(t) \geq 0 \), there is some \( t_2 \) with \( 0 < t_2 < t_1 \) and \( y'(t_2) > -\epsilon \). By the Mean Value Theorem, there is some \( t_3 \) with \( t_2 < t_3 < t_1 \) and \( \frac{y'(t_1) - y'(t_2)}{t_1 - t_2} = y''(t_3) \geq a(t_3)y(t_3) \geq 0 \). It follows that \( y'(t_1) \geq y'(t_2) > -\epsilon \).
Here’s a higher order generalization, using only the Mean Value Theorem, not the maximum value.

**Lemma 2.3.** Let \( k \geq 2 \) be an integer. Suppose \( a(t) \) is a real function on \([0, 1]\) such that \( a(t) \geq 0 \) and \( a \) is bounded on every subinterval \([0, x] \subseteq [0, 1]\). If \( y \) is continuous on \([0, 1]\) with \( y(0) \geq 0 \), and \( \lim_{t \to 0^+} y^{(j)}(t) \geq 0 \) for \( j = 1, \ldots, k - 1 \), and \( y^{(k)}(t) \geq a(t)y(t) \) for \( 0 < t < 1 \), then \( y(t) \geq 0 \) for all \( 0 \leq t \leq 1 \) and \( y^{(j)}(t) \geq 0 \) for all \( 0 < t < 1 \), \( j = 1, \ldots, k \).

**Proof.** Case 1: \( y(0) > 0 \) and \( \lim_{t \to 0^+} y^{(j)}(t) > 0 \). In this case, we can show that \( y > 0 \) on \([0, 1]\) and \( y^{(j)} > 0 \) on \((0, 1)\) for \( j = 1, \ldots, k - 1 \).

By Lemma 2.1 applied to \( y \) on \([0, 1]\), either \( y > 0 \) on \([0, 1]\), or there is some \( t_1 \in (0, 1) \) so that \( y(t) > 0 \) for all \( t \in [0, t_1) \) and \( y(t_1) = 0 \).

In the latter case, \( y(t_1) = 0 < y(0) \), so by the Mean Value Theorem for \( y \) on \([0, t_1]\), there is some \( t_2 \) with \( 0 < t_2 < t_1 \) and \( y'(t_2) < 0 \). Then, the MVT applies to \( y' \) on \([t_3, t_2]\) for some \( t_3 > 0 \) where \( y'(t_3) > 0 \), using \( \lim_{t \to 0^+} y'(t) > 0 \), so there is some \( t_4 > 0 \) where \( y''(t_4) = \frac{y'(t_3) - y'(t_2)}{t_3 - t_2} < 0 \). Repeatedly applying this MVT argument to \( y^{(j)} \) until \( j = k \), gives some \( t_N \) with \( 0 < t_N < t_1 \), \( y^{(k)}(t_N) < 0 \), contradicting \( y^{(k)}(t_N) \geq a(t_N)y(t_N) \geq 0 \).

So, the only case not leading to a contradiction is that \( y(t) > 0 \) on \([0, 1]\).

The above MVT argument also shows that all \( y^{(j)} \) are positive on \((0, 1)\) for \( j = 1, \ldots, k - 1 \), since any point \( t_n \) with \( 0 < t_n < t_1 = 1 \) and \( y^{(j)}(t_n) \leq 0 \) leads to another point \( t_m \) with \( 0 < t_m < t_n \) and \( y^{(j+1)}(t_m) < 0 \), eventually contradicting \( y^{(k)}(t_N) \geq a(t_N)y(t_N) \geq 0 \).

Case 1 did not use the boundedness of \( a \), just \( a \geq 0 \).

Case 2: \( y(0) \geq 0 \) and \( \lim_{t \to 0^+} y^{(j)}(t) \geq 0 \), \( j = 1, \ldots, k - 1 \). Suppose, toward a contradiction, that there is some \( t_0 \) with \( 0 < t_0 < 1 \) and \( y(t_0) < 0 \). For \( 0 \leq t \leq t_0 \), there is a bound \( A > 0 \) with \( 0 \leq a(t) \leq A \). For \( t \in [0, t_0] \), define \( u(t) = y(t) - \frac{1}{A} y(t_0) e^A(t-t_0) \). Then, by construction, \( u(0) > 0 \) and...
\[ u(t_0) < 0. \] For \( 0 < t < t_0, \ 1 \leq j \leq k - 1, \)
\[
\begin{align*}
u^{(j)}(t) & = y^{(j)}(t) - \frac{1}{2} y(t_0) A^{j/k} e^{A^{1/k} (t-t_0)} \\
& \quad \quad \Rightarrow \quad \lim_{t \to 0^+} u^{(j)}(t) = \left( \lim_{t \to 0^+} y^{(j)}(t) \right) - \frac{1}{2} y(t_0) A^{j/k} > 0,
\end{align*}
\]
\[
\begin{align*}
u^{(k)}(t) & = y^{(k)}(t) - \frac{1}{2} y(t_0) A e^{A^{1/k} (t-t_0)} \\
& \quad \geq a(t) y(t) - \frac{1}{2} y(t_0) a(t) e^{A^{1/k} (t-t_0)} \\
& \quad \quad + \frac{1}{2} y(t_0) a(t) e^{A^{1/k} (t-t_0)} - \frac{1}{2} y(t_0) A e^{A^{1/k} (t-t_0)} \\
& \quad \geq a(t) u(t) + \frac{1}{2} y(t_0) (a(t) - A) e^{A^{1/k} (t-t_0)} \\
& \quad \geq a(t) u(t).
\end{align*}
\]

Let \( w(t) = u(t_0 t), \) just horizontally re-scaling \( u \) to the domain \([0, 1]\) so that \( w(0) > 0, \ w(1) < 0, \ \lim_{t \to 0^+} w^{(j)}(t) > 0, \) and \( w^{(k)}(t) \geq t^k_0 a(t_0 t) w(t), \) so Case 1 applies to \( w, \) contradicting \( w(1) < 0. \) The conclusion is that \( y(t) \geq 0 \) on \([0, 1]\), and on \([0, 1]\) by continuity.

To establish the inequalities \( y^{(j)} \geq 0, \) start with \( j = k - 1. \) Then, for any \( t_1 \) with \( 0 < t_1 < 1 \) and any \( \epsilon > 0, \) from \( \lim_{t \to 0^+} y^{(k-1)}(t) \geq 0, \) there is some \( t_2 \) with \( 0 < t_2 < t_1 \) and \( y'(t_2) > -\epsilon. \) By the MVT, there is some \( t_3 \) with \( t_2 < t_3 < t_1 \) and \( \frac{y^{(k-1)}(t_1) - y^{(k-1)}(t_2)}{t_1 - t_2} = y^{(k)}(t_3) \geq a(t_3) y(t_3) \geq 0. \) It follows that \( y^{(k-1)}(t_1) \geq y^{(k-1)}(t_2) > -\epsilon. \) A similar argument applies for \( j \) decreasing from \( k - 1 \) to \( 1. \)

**Lemma 2.4.** If the left-side limit \( \lim_{t \to b^-} f(t) = -\infty, \) then there is no interval \((b - \delta, b)\) on which \( f'(t) \) is bounded below.

**Proof.** (See [C].)
Lemma 2.5. Suppose \( a(t) \) is a real function on \([0, 1]\) such that \( a \) is bounded above on every subinterval \([0, x] \subseteq [0, 1]\) and bounded below on every subinterval \([x_1, x_2] \subseteq (0, 1)\). If \( y \) is continuous on \([0, 1]\) with \( y(0) \geq 0 \) and \( y'(t) \geq a(t)y(t) \) for \( 0 < t < 1 \), then \( y(t) \geq 0 \) for all \( 0 \leq t \leq 1 \).

Proof. Case 1: \( y(0) > 0 \).

By Lemma 2.1 applied to \( y \) on \([0, 1]\), either \( y > 0 \) on \([0, 1]\), or there is some \( t_1 \in (0, 1) \) so that \( y(t) > 0 \) for all \( t \in [0, t_1) \) and \( y(t_1) = 0 \). If \( t_1 = 1 \), then \( y \geq 0 \) as claimed.

So, suppose toward a contradiction that \( t_1 < 1 \). On the interval \([\frac{1}{2}t_1, t_1]\), \( a \) is bounded below: there is some \( K < 0 \) so that \( K < a(t) \). Define the function \( f(t) = \ln(y(t)) \) for \( t \) in the interval \((\frac{1}{2}t_1, t_1)\). \( f \) has left-side limit \( \lim_{t \to t_1^-} f(t) = -\infty \). For all \( t \in (\frac{1}{2}t_1, t_1) \), the derivative is bounded below: \( f'(t) = \frac{1}{y(t)}y'(t) \geq \frac{1}{y(t)}a(t)y(t) = a(t) \geq K \), but this contradicts Lemma 2.4.

Case 2: \( y(0) = 0 \).

Suppose toward a contradiction that there is some \( p \in [0, 1] \) with \( y(p) < 0 \). \( p \neq 0 \) by assumption, and if \( p = 1 \), then by continuity of \( y \), there is some nearby point \( p - \delta_1/2 \) with \( y(p - \delta_1/2) < 0 \). So by re-labeling if necessary, we can assume \( 0 < p < 1 \). On the interval \([0, p]\), \( a \) is bounded above: there is some \( A > 0 \) so that \( a(t) \leq A \).

Define \( g(t) = -y(p - pt) \) on the domain \( 0 \leq t \leq 1 \), so that \( g(0) = -y(p) > 0 \), \( g(1) = -y(0) = 0 \), and \( g \) is continuous on \([0, 1]\). The derivative satisfies

\[
g'(t) = py'(p - pt) \geq pa(p - pt)y(p - pt) = -pa(p - pt)g(t),
\]

and the coefficient \(-pa(p - pt)\) is bounded below by \(-pA\). Case 1 applies to \( g \), so \( g(t) > 0 \) on \([0, 1]\) and \( g(1) = 0 \). Define the function \( f(t) = \ln(g(t)) \) for \( t \) in the interval \((0, 1)\). \( f \) has left-side limit \( \lim_{t \to 1^-} f(t) = -\infty \). For all \( t \in (0, 1) \), the derivative is bounded below: \( f'(t) = \frac{1}{g(t)}g'(t) \geq \frac{1}{g(t)}(-pa(p - pt))g(t) = -pa(p - pt) \geq -pA \), but this contradicts Lemma 2.4. □
Lemma 2.6. Suppose $a(t)$ is a bounded real function on $[0, X]$. Then there is some $\delta$ with $0 < \delta \leq X$, with the property that if $y$ is continuous on $[0, X]$ with $y(0) \geq 0$, $\lim_{t \to 0^+} y(t) \geq 0$, and $y''(t) \geq a(t)y(t)$ for $0 < t < X$, then $y(t) \geq 0$ for all $0 \leq t \leq \delta$.

Proof. Step 1. Pick any $x$ in $(0, X]$, so that by hypothesis, there are some $A > 0$ and $K < 0$ so that $K \leq a(t) \leq A$ for $t \in [0, x]$. Let $\delta = \min\{x, \frac{1}{\sqrt{A}}, \frac{1}{\sqrt{K}}\} > 0$. (Remark: depending on $a$, it may be possible to choose $x$ that optimizes $\delta$.) To show that this is a $\delta$ as claimed by the Lemma, suppose toward a contradiction that there is some $c$ with $0 \leq c \leq \delta$ and $y(c) < 0$. $c > 0$ by hypothesis.

Step 2. Let $y(b)$ be the minimum value of $y$ on $[0, c]$, so $y(b) \leq y(c) < 0$ and $0 < b \leq c \leq \delta$. The MVT applies to $y$ on $[0, b]$; there is some $t_0$ with $0 < t_0 < b$ and $y'(t_0) = \frac{y(b) - y(0)}{t_0 - 0}$. The MVT applies to $y'$ (extended to $y'(0) = \lim_{t \to 0^+} y'(t)$) on $[0, t_0]$; there is some $t_1$ with $0 < t_1 < t_0$ and

$$y''(t_1) = \frac{y'(t_0) - y'(0)}{t_0 - 0} = \frac{y(b) - y(0)}{t_0} - y'(0).$$

By hypothesis,

$$a(t_1)y(t_1) \leq y''(t_1) = \frac{y(b) - y(0) - by'(0)}{bt_0} \leq \frac{y(b)}{bt_0} < 0.$$

If $a(t_1) > 0$ and $y(t_1) < 0$, then $y(t_1) \leq \frac{y(b)}{bt_0a(t_1)} < \frac{y(b)}{bt_0} \leq y(b)$, contradicting the minimum property of $y(b)$. So, $a(t_1) < 0$ and $y(t_1) > 0$.

Step 3. Lemma 2.1 applies to $y$ on the interval $[t_1, b]$, so there is some $t_3$ with $t_1 < t_3 < b$, $y(t_3) = 0$, and $y(t) > 0$ for all $t \in [t_1, t_3)$. A left-side version of Lemma 2.1 applies to $y$ on the interval $[0, t_1]$; there are two cases:

Case 1. There is some $t_2$ with $0 \leq t_2 < t_1$, $y(t_2) = 0$, and $y(t) > 0$ for all $t \in (t_2, t_1]$.

Case 2. $y(t) > 0$ for all $t \in [0, t_1]$. In this case denote $t_2 = 0$.

In either case, there is some interval $[t_2, t_3]$ where $0 \leq t_2 < t_1 < t_3 < b$, $y(t_3) = 0$, $y(t_2) \geq 0$, and $y(t) > 0$ for all $t \in (t_2, t_3)$. Let $y(t_4)$ be the maximum value of $y$ on $[t_2, t_3]$. In Case 1, $t_2 < t_4 < t_3$, so the maximum occurs at an interior point and $y'(t_4) = 0$. In Case 2, $t_4$ is either an interior point of $[t_2, t_3]$, or the maximum occurs at the endpoint $t_4 = t_2 = 0$, where there is a right-side derivative $y'(0) \geq 0$ as in Step 2. In either case, $y'(t_4) \geq 0$. 

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Step 4. The MVT applies to $y$ on $[t_4, t_3]$: there is some $t_5$ with $t_4 < t_5 < t_3$ and $y'(t_5) = \frac{y(t_5) - y(t_4)}{t_5 - t_4}$. The MVT applies to $y'$ on $[t_4, t_5]$; there is some $t_6$ with $t_4 < t_6 < t_5$ and

$$y''(t_6) = \frac{y'(t_5) - y'(t_4)}{t_5 - t_4} = \frac{y(t_5) - y(t_4)}{t_5 - t_4} - y'(t_4) < \frac{-y(t_4)}{b^2}.$$  

Using the lower bound for $a$ and the property $y(t_6) > 0$,

$$K y(t_6) \leq a(t_6) y(t_6) \leq y''(t_6) \leq \frac{-y(t_4)}{b^2} \implies y(t_6) > \frac{-y(t_4)}{b^2 K} \geq \frac{-y(t_4)}{\delta^2 K} \geq y(t_4),$$

contradicting the maximum property of $y(t_4)$.

**Theorem 2.7.** Suppose $a(t)$ and $b(t)$ are real functions on $[0,1)$, and there is a point $X$ such that $0 < X < 1$ and $a$, $b$, and $b'$ are bounded on $(0, X]$. Then there is some $\delta$ with $0 < \delta \leq X$, with the property that if $y$ is continuous on $[0,1)$ with $y(0) \geq 0$, $\lim_{t \to 0^+} y'(t) \geq 0$, and $y''(t) \geq a(t)y(t) + b(t)y'(t)$ for $0 < t < X$, then $y(t) \geq 0$ for all $0 \leq t \leq \delta$.

**Proof.** By hypothesis, there are some $A > 0$ and $K < 0$ so that $K \leq a(t) \leq A$ for $t \in [0, X]$, and there are some $B > 0$ and $L < 0$ so that $L \leq b(t) \leq B$ for $t \in [0, X]$. Let $x = \min\{X, \frac{1}{\sqrt{2A}}, \frac{1}{4B}\} > 0$.

Recall the elementary calculus fact that if $b$ is continuous and bounded on $(0, x)$, then $b$ is (Riemann) integrable on $[0, x]$. Let $p(t) = \int_0^t \frac{1}{2} b(x) \, dx$, so $p$ is continuous on $[0, x]$ and for $0 < t < x$, $p'(t) = \frac{1}{2} b(t)$.

Let

$$f(t) = e^{p(t)} \left[ y(t) - Ky(0)t^2 - y(0) \right],$$

so $f$ is continuous on $[0, x]$ and $f(0) = 0$. For $0 < t < x$,

$$f'(t) = e^{p(t)} \left[ y'(t) - 2Ky(0)t \right]$$

$$+ e^{p(t)} \left( -\frac{1}{2} b(t) \right) \left[ y(t) - Ky(0)t^2 - y(0) \right],$$

and using $\lim_{t \to 0^+} (y(t) - y(0)) = 0$ and the boundedness of $b$, the limit exists:

$$\lim_{t \to 0^+} f'(t) = \lim_{t \to 0^+} y'(t) \geq 0.$$
For $0 < t < x$,

\[
f''(t) = e^{p(t)} [y''(t) - 2K y(0)] + e^{p(t)} \left( -\frac{1}{2} b(t) \right) \left[ y'(t) - 2K y(0) t \right]
\]

\[
+ e^{p(t)} \left( -\frac{1}{2} b(t) \right)^2 \left[ y(t) - K y(0) t^2 - y(0) \right]
\]

\[
+ e^{p(t)} \left( -\frac{1}{2} b'(t) \right) \left[ y(t) - K y(0) t^2 - y(0) \right]
\]

\[
+ e^{p(t)} \left( -\frac{1}{2} b(t) \right) \left[ y'(t) - 2K y(0) t \right]
\]

\[
\geq e^{p(t)} \left[ a(t) y(t) + b(t) y'(t) - 2K y(0) \right] - e^{p(t)} b(t) \left[ y'(t) - 2K y(0) t \right]
\]

\[
+ e^{p(t)} \left( -\frac{1}{2} b(t) \right)^2 \left[ y(t) - K y(0) t^2 - y(0) \right]
\]

\[
+ e^{p(t)} \left( -\frac{1}{2} b'(t) \right) \left[ y(t) - K y(0) t^2 - y(0) \right]
\]

\[
= e^{p(t)} \left[ y(t) - K y(0) t^2 - y(0) \right] \left( a(t) + \frac{1}{4} (b(t))^2 - \frac{1}{2} b'(t) \right)
\]

\[
+ e^{p(t)} y(0) \left( K (a(t) t^2 + 2b(t) t - 2) + a(t) \right). 
\]

In the last step, the $y'$ terms cancel by construction. Term (4) is equal to $\tilde{a}(t) f(t)$, where $\tilde{a}(t) = \left( a(t) + \frac{1}{4} (b(t))^2 - \frac{1}{2} b'(t) \right)$ is bounded by hypothesis. The upper bounds $a(t) \leq A$ and $b(t) \leq B$ and the initial choice of $x$ imply, for $0 < t < x$,

\[
a(t) t^2 + 2b(t) t - 2 \leq A t^2 + 2B t - 2
\]

\[
\leq A \left( \frac{1}{2A} \right) + 2B \left( \frac{1}{4B} \right) - 2 = -1
\]

\[
\implies K (a(t) t^2 + 2b(t) t - 2) + a(t) \geq -K + a(t) \geq 0,
\]

so the entire term (5) is non-negative, and for $0 < t < x$, $f''(t) \geq \tilde{a}(t) f(t)$. Lemma 2.6 applies to $f$, so there is some $\delta_1$ depending on $a$, $b$, $b'$, $X$, but not on $y$, with $f \geq 0$ on $[0, \delta]$. The factor $[y(t) - K y(0) t^2 - y(0)]$ is non-negative on the same interval, where

\[
y(t) - K y(0) t^2 - y(0) \geq 0 \implies y(t) \geq y(0)(1 + K t^2),
\]

so $y(t) \geq 0$ for $0 \leq t \leq \delta = \min\{\delta_1, \sqrt{-K}\}$. 

\[\blacksquare\]
2.2 A nonlinear differential inequality

**Theorem 2.8.** Given a function \( f \) that satisfies \( f'' f - (f')^2 \geq 0 \) on \((a,b)\), at every critical point \( c \) with \( f(c) \neq 0 \), there is either a positive local min. or a negative local max.

**Proof.** Suppose \( c \) is a critical point, meaning \( f'(c) = 0 \). Suppose also that \( f(c) \neq 0 \), so that the function \( g(x) = f'(x)/f(x) \) is defined on a neighborhood \( N = (c-\delta, c+\delta) \subseteq (a,b) \). By the quotient rule,

\[
g'(x) = \frac{f(x)f''(x) - (f'(x))^2}{(f(x))^2}
\]

which is \( \geq 0 \) on \( N \) by hypothesis. It follows that \( g(x) \) is weakly increasing on \( N \). For \( c < x < c + \delta \), \( f'(x)/f(x) = g(x) \geq g(c) = 0 \), and for \( c - \delta < x < c \), \( f'(x)/f(x) = g(x) \leq g(c) = 0 \).

If \( f(c) > 0 \), then \( f(x) > 0 \) on \( N \) so \( f'(x) \geq 0 \) on the right and \( f'(x) \leq 0 \) on the left. \( f(c) \) is a local min. by the first derivative test.

If \( f(c) < 0 \), then \( f(x) < 0 \) on \( N \) so \( f'(x) \leq 0 \) on the right and \( f'(x) \geq 0 \) on the left. \( f(c) \) is a local max.

Note that \( C^2 \) is not used, just the existence of \( f'' \). Constant functions trivially satisfy both the hypotheses and conclusions.

**Lemma 2.9.** If \( p(x) \) satisfies \( p''(x) \geq 0 \) on \((a,b)\) then for any \( c \in (a,b) \), \( p \) satisfies \( p(x) \geq p(c) + p'(c)(x-c) \) for all \( x \in (a,b) \).

**Proof.** (See [C].)

**Theorem 2.10.** Given a function \( f \) that satisfies \( f'' f - (f')^2 \geq 0 \) on \((a,b)\), if there is a point \( c \) in \((a,b)\) with \( f(c) > 0 \), then \( f \) satisfies

\[
f(x) \geq f(c) \cdot \exp\left(\frac{f'(c)}{f(c)}(x-c)\right)
\]

for all \( x \in (a,b) \).

**Proof.** By continuity, there is some neighborhood \( (s,t) \subseteq (a,b) \) so that \( s < c < t \) and \( f(x) > 0 \) on \((s,t)\). Suppose \( f(z) = 0 \) for some \( z \in (c,b) \). Then, the set \( \{t : f(t) > 0 \text{ on } (c,t)\} \) is non-empty and has \( \text{sup} = T \leq z < b \). By construction and using continuity again, \( f(T) = 0 \) and \( f(x) > 0 \) on \((s,T)\).
Consider $h(x) = \ln(f(x))$, which is well-defined on $(s, T)$. $h' = f' / f = g$, from Theorem 2.8, so $h'' = g' \geq 0$ on $(s, T)$. By Lemma 2.9, $h(x) \geq h(c) + h'(c) \cdot (x - c)$ on $(s, T)$:

\[
\ln(f(x)) \geq \ln(f(c)) + \frac{f'(c)}{f(c)} \cdot (x - c)
\]

\[
f(x) \geq f(c) \cdot \exp\left(\frac{f'(c)}{f(c)} \cdot (x - c)\right)
\]

for all $x$ in $(s, T)$. This implies $\lim_{x \to T^-} f(x) = f(T) > 0$, which contradicts the construction of $T$. We can conclude that $f$ is never zero on $(c, b)$, and always positive there, so the inequality holds on $(s, b)$. The inequality on the other side of $c$ follows from an analogous inf argument.

It follows that if $c$ is a critical point with $f(c) > 0$, then $f(c)$ is a global minimum. It further follows that there is at most one such critical point.

## References