The Attraction of Surfaces of Revolution

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In my lectures for the first-year calculus sequence, I state and solve physics problems. After the section on surface area, the following problem generated some interest:

Assuming an inverse square law of attraction, what is the force exerted by a massive surface of revolution on a point mass \( m \) located on the axis of symmetry?

An important special case is the attractive force of gravity exerted by a spherical shell on a point mass \( m \). Since any line through the center is an axis of symmetry, \( m \) can be anywhere in space.

For the general case, here are some preliminary assumptions:

1. The surface of revolution is defined by a nonnegative function \( f(x) \) on a closed interval \([a, b] \), such that \( f' \) exists on \((a, b) \). The graph of \( f \) is revolved around the \( x \)-axis as in the Figure.

2. The surface’s mass is distributed evenly, in the sense that it has a constant “planar density,” \( d \geq 0 \). The units on \( d \) might be kilograms per square meter, for example, to distinguish it from linear or spatial density.

3. The “inverse square law” refers to a force exerted on a point mass \( m \) by another point mass \( M \) separated by distance \( r > 0 \). Then the magnitude of the force is \( GmMr^{-2} \), for a positive constant \( G \). \( M, m \) will be assumed nonnegative, and the direction of the force on \( m \) is toward \( M \).

4. To simplify calculation, the point mass \( m \) can be assumed to be at the origin, by translating \( f \) left or right if necessary.
To start with the solution to the physics problem, we slice the surface with planes parallel to the yz-plane, and review a (sketchy) derivation of the integral formula for surface area.

The Riemann sum procedure is to partition $[a, b]$ into $n$ subintervals $[x_{i-1}, x_i]$, with length $\Delta x_i$, for $i = 1$ to $n$, and then select a sample point $x_i^*$ — the midpoint will be convenient. The graph of $f$ can be approximated by $n$ line segments $L_i$ connecting $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$. Revolving each segment $L_i$ gives a truncated cone $C_i$, which approximates a slice $S_i$ of the surface of revolution. Each $C_i$ has surface area

$$\pi(f(x_{i-1}) + f(x_i)) \sqrt{\Delta x_i^2 + (\Delta f_i)^2},$$

where $\Delta f_i$ abbreviates $f(x_i) - f(x_{i-1})$. (This well-known formula for the area of a truncated cone can be derived without calculus.) The average $\frac{1}{2}(f(x_{i-1}) + f(x_i))$ is the distance from $(x_i^*, 0)$ to the midpoint of $L_i$, which can be approximated by $f(x_i^*)$. Then, the approximate area of $C_i$, and the slice $S_i$, is $2\pi f(x_i^*) \sqrt{1 + \left(\frac{M_i}{\Delta x_i}\right)^2} \Delta x_i$, and the mass of $S_i$, denoted $M_i$, is
approximately the density times this area:

\[ M_i \approx d \cdot 2\pi f(x_i^*) \sqrt{1 + \left( \frac{\Delta f_i}{\Delta x_i} \right)^2 \Delta x_i}. \]

The total area of the surface is the \( n \to \infty \) (and \( \max \Delta x_i \to 0 \)) limit of the sum of the approximate areas, and its total mass, denoted \( M(f) = \sum M_i \), is equal to \( d \) times this area:

\[ M(f) = d \cdot \int_a^b 2\pi f(x) \sqrt{1 + \left( \frac{df}{dx} \right)^2} \, dx. \]

The force \( F_i \), exerted by each slice \( S_i \) on the mass \( m \) at the origin, will be directed along the \( x \)-axis. This is obvious by the rotational symmetry, and also follows from the following approximation of \( F_i \) as a vector sum. The slice \( S_i \) can itself be subdivided "radially" into \( 2N \) pieces by \( N \) planes through the \( x \)-axis. When \( n \) and \( N \) are large, each of these pieces can be treated as a point with mass \( M_i / 2N \), for the purposes of approximating \( F_i \) using the inverse square law. Every piece of \( S_i \) will be represented by one of its points, with horizontal coordinate \( x_i^* \) and at distance \( f(x_i^*) \) from the \( x \)-axis, so that the line from \( m \) to this point is at an angle \( \theta_i \) with the \( x \)-axis. The force exerted on \( m \) by the piece has approximate magnitude \( Gm M_i / (x_i^*)^2 + (f(x_i^*))^2 \). Its horizontal component (along the \( x \)-axis) is \( \cos(\theta_i) \) times the magnitude, and its radial component is \( \sin(\theta_i) \) times the magnitude. The force exerted by the opposite piece (rotating the piece and its representative point by \( 180^\circ \)) has the same horizontal component, but an oppositely directed radial component. In the sum over \( 2N \) pieces, the radial components all cancel, and the approximate horizontal components total to

\[
F_i \approx Gm \frac{M_i}{(x_i^*)^2 + (f(x_i^*))^2} \cos(\theta_i)
\]

\[
\approx Gm \frac{d2\pi f(x_i^*)}{(x_i^*)^2 + (f(x_i^*))^2} \sqrt{1 + \left( \frac{\Delta x_i}{\Delta x_i} \right)^2} \Delta x_i \]

\[
= 2\pi Gm d \frac{x_i^* f(x_i^*)}{((x_i^*)^2 + (f(x_i^*))^2)^{3/2}} \Delta x_i.
\]

The second step uses the earlier approximation for \( M_i \), and the ratio for the cosine: \( \cos(\theta_i) = \frac{x_i^*}{\sqrt{(x_i^*)^2 + (f(x_i^*))^2}} \). This formula for \( F_i \) is actually a signed
quantity, with the formula for the cosine taking into account the direction of the force acting on \( m \): to the right for \( x_i^+ > 0 \), and to the left for \( x_i^- < 0 \).

So, in the \( n \to \infty \) limit, the answer to the physics question is

\[
\int_a^b 2\pi Gmd \frac{xf(x)\sqrt{1 + (f'(x))^2}}{x^2 + (f(x))^2} \, dx,
\]

assuming that this definite integral exists, which (mathematically) is a non-trivial condition required of \( f \).

As an application of this formula, consider a sphere with center \((c,0)\) (on the positive \( x \)-axis, \( c > 0 \)) and radius \( R > 0 \). Using the above formula, with \( f(x) = \sqrt{R^2 - (x - c)^2} \), and \([a, b] = [c - R, c + R]\) gives \( f'(x) = \frac{c - x}{\sqrt{R^2 - (x - c)^2}} \), and total force

\[
F = \int_{c-R}^{c+R} 2\pi Gmd \frac{x\sqrt{R^2 - (x - c)^2}\sqrt{1 + \left(\frac{c - x}{\sqrt{R^2 - (x - c)^2}}\right)^2}}{x^2 + R^2 - (x - c)^2} \, dx
\]

\[
= 2\pi GmdR \int_{c-R}^{c+R} \frac{x}{(R^2 + 2xc - c^2)^{3/2}} \, dx
\]

\[
= 2\pi GmdR \left. \frac{c^2}{c^2 + x^2} \right|_{c-R}^{c+R}
\]

\[
= \begin{cases} 
2\pi Gmd & \text{if } c \neq R \\
4\pi GmdR^2 c^{-2} & \text{if } c > R \\
0 & \text{if } c < R \\
2\pi Gmd & \text{if } c = R.
\end{cases}
\]

The total mass of the sphere is \( M(f) = 4\pi R^2 d \), and if this mass were concentrated at the center \((c,0)\) with \( c > R \), the force on the mass \( m \) at \((0,0)\) would be \( GmM(f)c^{-2} \). This is the same as the above integral, so we have a single-variable derivation of a result of Newton, that the external gravitational attraction of a sphere is equal to the attractive force of a point with the same mass at the sphere’s center. This was part of Newton’s argument that a solid ball has the same property.

The same integral also demonstrates the fact that if the particle of mass \( m \) is inside the sphere, so \( c < R \), then it feels no force acting in any direction. (This fact was interesting and surprising to many students.) At \( c = R \), the
particle is on the sphere, and the force is \( \frac{1}{2} G m M f(c^{-2}) \); plotting \( F \) as a function of \( c \), there is a discontinuity at \( c = R \). The \( c = 0 \) and \( c < 0 \) cases follow from similar calculations.

Other surfaces of revolution for which the above integral formula might be tractable are cylinders, \( f(x) = K \), truncated cones, \( f(x) = kx + K \), or funnel shapes, \( f(x) = k/x \), over intervals where \( f(x) \geq 0 \). The construction also could be applied to a repelling force.