

You should draw a picture, where appropriate.

1. Prove *de Morgan's law*

**Lemma 3.1** Let  $\{S_\alpha\}_{\alpha \in A}$  be a collection of subsets in  $\mathbf{R}^n$ . Then

$$\cup_{\alpha \in A} S_\alpha = c \left[ \cap_{\alpha \in A} (cS_\alpha) \right], \quad \cap_{\alpha \in A} S_\alpha = c \left[ \cup_{\alpha \in A} (cS_\alpha) \right].$$

**Proof:** We shall prove the first of the equalities, the other being similar. Let  $x \in \cup_{\alpha \in A} S_\alpha$ . Then  $x \in S_\alpha$  for some fixed  $\alpha$ . But then  $x \notin c(S_\alpha)$  for this choice of  $\alpha$ . Therefore,  $x \notin \cap_{\alpha \in A} (cS_\alpha)$  by the definition of intersection. But then  $x \in c \left[ \cap_{\alpha \in A} (cS_\alpha) \right]$  by the definition of a complement. So

$$\cup_{\alpha \in A} S_\alpha \subseteq c \left[ \cap_{\alpha \in A} (cS_\alpha) \right].$$

We now prove the inclusion in the opposite direction. Let  $x \in c \left[ \cap_{\alpha \in A} (cS_\alpha) \right]$ . Then  $x \notin \cap_{\alpha \in A} (cS_\alpha)$ , and hence  $x \notin c(S_\alpha)$  for some fixed  $\alpha$  (otherwise,  $x$  would be an element of the intersection). The last conclusion implies that  $x \in S_\alpha$  for this particular  $\alpha$ , and therefore it belongs to the union  $\cup S_\alpha$ , which proves the inclusion in the opposite direction. This implies the Lemma.  $\square$

2. Show that for  $\mathbf{x} \in \mathbf{R}^n$  and  $r > 0$  the set  $B(\mathbf{x}, r)$  is open; that is, show that an open ball is open.

**Proof:** Let  $\mathbf{y} \in B(\mathbf{x}, r)$  be arbitrary. Then  $\|\mathbf{y}\| < r$  and so  $\epsilon := r - \|\mathbf{y}\| > 0$ . We shall show that  $B(\mathbf{y}, \epsilon) \subseteq B(\mathbf{x}, r)$ , which will prove that the open ball is an open set. Indeed, let  $\mathbf{w} \in B(\mathbf{y}, \epsilon)$ . Then  $\|\mathbf{w} - \mathbf{y}\| < \epsilon$ , and therefore

$$\|\mathbf{w} - \mathbf{x}\| \leq \|\mathbf{w} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{x}\| < \epsilon - \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{y} - \mathbf{x}\| = r. \quad \square$$

3. Let  $A$  and  $B$  be two bounded sets of real numbers with  $A \subseteq B$ . Show that

$$\sup\{a : a \in A\} \leq \sup\{b : b \in B\}, \quad \inf\{a : a \in A\} \geq \inf\{b : b \in B\}.$$

**Proof:** For brevity denote the l.u.b. of a set  $G$  with  $\sup G$ . We have to show that  $\sup A \leq \sup B$ . Since  $\sup B$  is an upper bound of  $B$ , we have that  $\sup B \geq b$  for all  $b \in B$ . From the hypothesis  $A \subseteq B$ , so  $\sup B \geq a$  for all  $a \in A$ , i.e.  $\sup B$  is an upper bound for  $A$ . By the definition of l.u.b. we get that  $\sup A \leq \sup B$ .  $\square$