Powers of 10

Exponential Notation

Chemists often work with very large or very small numbers which are cumbersome in mathematical calculations. The speed of light for example is 299,792,458 m/s and the mass of a single carbon atom is 0.000 000 000 000 000 000 000 019 93 g. Calculations which involve these numbers are greatly simplified through the use of exponential notation in which a number is expressed as some quantity greater than or equal to 1, but less than 10 multiplied by 10 raised to some power.

Before we continue we should review some basic mathematical properties.

(1) Any number (except 0) raised to the zeroth power is equal to 1.

$1^0 = 1 \quad 10^0 = 1 \quad x^0 = 1$ etc.

(2) Any number raised to the first power is equal to itself.

$1^1 = 1 \quad 8^1 = 8 \quad 10^1 = 10$ etc.

(3) A number raised to the second power is equal to the product of that number times itself.

$1^2 = 1 \quad 2^2 = 4 \quad 10^2 = 100$ etc.

(4) A number raised to the nth power is the product of that number times itself $n - 1$ times.

$2^4 = 2 \times 2 \times 2 \times 2 = 16 \qquad 10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$

(5) Division by a number raised to some exponent is equivalent to multiplying by that number raised to an exponent of the opposite sign.

$$\frac{1}{10^2} = 1 \times 10^{-2} \qquad \frac{3}{4^{-3}} = 3 \cdot \frac{1}{4^{-3}} = 3 \times 4^3$$

These rules form the basis for writing numbers in exponential notation, as a number between 1 and 10 which is then multiplied by 10 raised to some exponent. For example,

$$1645 = 1.645 \times 1000 = 1.645 \times 10 \times 10 \times 10 = 1.645 \times 10^3$$

$$34.2 = 3.42 \times 10 = 3.42 \times 10^1$$

For values smaller than 1, it might be helpful to write the decimal as its fractional equivalent, and then make the numerator a value between 1 and 10.

$$0.015 = \frac{15}{1000} = \frac{15}{1000} \cdot \frac{\left(\frac{1}{10}\right)}{\left(\frac{1}{10}\right)} = \frac{1.5}{100} = \frac{1.5}{10^2} = \frac{1.5}{10^2} = 1.5 \times 10^{-2}$$

$$0.000269 = \frac{269}{100{,}000} = \frac{269}{100{,}000} \cdot \frac{\left(\frac{1}{100}\right)}{\left(\frac{1}{100}\right)} = \frac{2.69}{10{,}000} = \frac{2.69}{10^4} = 2.69 \times 10^{-4}$$

The power of ten used in exponential notation is the number of times that the decimal point must be moved to produce a number between 1 and 10. To convert 3,367,004 to exponential notation we must move the decimal point six times.

$$3{,}367{,}004 = 3.367004 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 3.367004 \times 10^6$$

To convert 0.098 to exponential notation we must move the decimal point twice.

$$0.098 = \frac{98}{1000} = \frac{98}{1000} \cdot \frac{\left(\frac{1}{10}\right)}{\left(\frac{1}{10}\right)} = \frac{9.8}{100} = \frac{9.8}{10^2} = 9.8 \times 10^{-2}$$

Most students rapidly learn to write numbers in exponential notation, but they still may have difficulty when asked to use this notation in calculations. We can readily understand how to manipulate numbers in this notation if we note that addition and subtraction can only be performed when the exponents of both numbers are the same.

$$(A \times 10^x) + (B \times 10^x) = (A + B) \times 10^x$$

whereas multiplication and division of exponential numbers with the same base can be performed by adding or subtracting their exponents.

$$(10^x) \times (10^y) = 10^{(x+y)}$$

$$\frac{10^x}{10^y} = 10^{(x-y)}$$

Thus,

$$1.05 \times 10^3 + 2.36 \times 10^3 = (1.05 + 2.36) \times 10^3 = 3.41 \times 10^3$$

but, adding 1.05×10^3 and 2.36×10^2 requires the conversion of one of these numbers to a different exponent.

$$1.05 \times 10^3 + 0.236 \times 10^3 = 1.29 \times 10^3$$

$$10.5 \times 10^2 + 2.36 \times 10^2 = 12.9 \times 10^2 = 1.29 \times 10^3$$

Note that the answer to this problem is not 1.286×10^3. The rules for **significant figures** (discussed below) will make it clear why 1.29×10^3 is the correct answer.

When multiplying two numbers in exponential notation we first multiply the coefficients and then add the exponents

$$(A \times 10^x) \times (B \times 10^y) = (A \times B) \times 10^{(x+y)}$$

$$(5.0 \times 10^3) \times (1.6 \times 10^2) = (5.0)(1.6)(10^3)(10^2) = (5.0)(1.6) \times 10^{(3+2)} = 8.0 \times 10^5$$

Similarly, in division we first divide the coefficients in the usual manner then subtract the exponents.

$$\frac{7.0 \times 10^3}{1.8 \times 10^2} = \frac{7.0}{1.8} \times 10^{(3-2)} = 3.9 \times 10^1$$

The most common sources of error in the manipulation of numbers in exponential notation occur when the result of a calculation is a number such as 2357.8×10^3. Is this equal to 2.3578×10^0 or is it equal to 2.3578×10^6 ? There are a number of ways in which this problem can be solved. The method we prefer involves the following line of thought.

Express 2357.8 in exponential notation and then multiply by 10^3.

$$2357.8 \times 10^3 = (2.3578 \times 10^3) \times 10^3 = 2.3578 \times 10^6$$

Similarly,

$$0.005623 \times 10^5 = (5.623 \times 10^{-3}) \times 10^5 = 5.623 \times 10^2$$

Another word of caution is appropriate here. When entering a number such as 10^3 into their calculators, students may incorrectly use the following keystrokes:

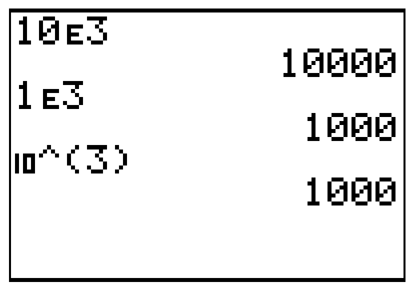
[1] [0] [2nd] [EE ,] [3]

These students are then puzzled when the result of the calculation is "off" by a factor of 10.

The command k [2nd] [EE ,] p performs the calculation $k \times 10^p$ for any numbers k and p.
The correct series of keystrokes would be

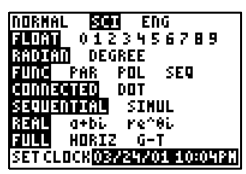
[1] [2nd] [EE ,] [3]

which gives the same result as [2nd] [10^x LOG] [3] [)]



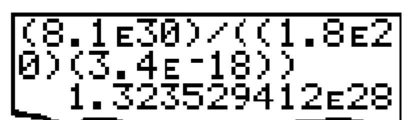
Calculations using the [2nd] [EE ,] feature of a graphing calculator combined with the SCI mode can be very helpful.

Press [MODE] and highlight SCI. Press ENTER.
Press [2nd] QUIT to return to the home screen.



To compute $\dfrac{(8.1\times10^{30})}{(1.8\times10^{20})(3.4\times10^{-18})}$

remember order of operations and type as on the screen to the right:
(Note that the use of parentheses is NOT optional.)



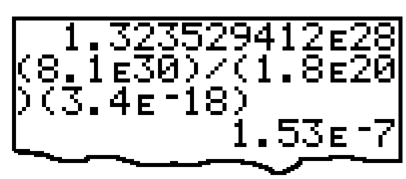
The calculator gives us more precision than we can use. We report 1.3×10^{28}. according to the rules of significant figures (discussed later). Without doing any calculations, we can estimate:

$$\frac{(8.1\times10^{30})}{(1.8\times10^{20})(3.4\times10^{-18})} = \frac{(8.1)}{(1.8)(3.4)}\times\frac{10^{30}}{10^{20}10^{-18}}$$

$$= \frac{8.1}{(1.8)(3.4)}\times\frac{10^{30}10^{18}}{10^{20}} = \frac{8.1}{(1.8)(3.4)}\times10^{10}10^{18} = \frac{8.1}{(1.8)(3.4)}\times10^{28}$$

If you were to incorrectly omit the parentheses, as on the screen to the left, we would be something other than what we expect.

In fact, the calculator is computing $\dfrac{(8.1\times10^{30})}{(1.8\times10^{20})}(3.4\times10^{-18})$.



Logarithms

Sometimes we can write functions as words. As an example, in some computer programs, to calculate $\sqrt{2}$ one types the command $\mathtt{sqrt}(2)$, which then returns the positive real number which, when squared, gives 2. Another example is the absolute value. To find $|-3|$ on a graphing calculator, one uses the command $\mathtt{abs}(-3)$. Throughout this course, it will be very useful to find the **exponent of 10** which makes a given number. One possibility to write this function in words is:

$$\mathtt{exponentof10whichmakes}(x), \text{ where } x \text{ is the given number.}$$

(We will see another possibility in a moment to write this function, but for now, consider this way.) Suppose we wanted to find what we raise 10 to in order to get 1000. Using this new function, we could write this as the $\mathtt{exponentof10whichmakes}(1000)$.
Since $1000 = 10^3$, we could then say

$$\mathtt{exponentof10whichmakes}(1000) = \boxed{\mathtt{exponentof10whichmakes}(10^3) = 3.}$$

How would we find $\mathtt{exponentof10whichmakes}(0.1)$? Since have $0.1 = 10^{-1}$, then

$$\mathtt{exponentof10whichmakes}(0.1) = \boxed{\mathtt{exponentof10whichmakes}(10^{-1}) = -1.}$$

Similarly, $\mathtt{exponentof10whichmakes}(\sqrt{10}) = \boxed{\mathtt{exponentof10whichmakes}(10^{1/2}) = ½.}$

What do you expect is true in general of $\mathtt{exponentof10whichmakes}(10^a)$?
Looking at the three above examples in the boxes, we can generalize that

(1) $\boxed{\mathtt{exponentof10whichmakes}(10^a) = a}$

The property in Equation (1) has a special name. It is called an inverse property, since once we take the exponent *a* and make it the power on the base 10 (also called "exponentiating" *a*), then the function $\mathtt{exponentof10whichmakes}(10^a)$ simply asks for what we started with: the exponent *a*.

What do you expect is true in general of $10^{\mathtt{exponentof10whichmakes}(a)}$?
It might help to look at examples first, in order to see any pattern.
$\mathtt{exponentof10whichmakes}(1000) = 3$, so $\mathbf{10}^{\mathtt{exponentof10whichmakes}(1000)} = \mathbf{10}^3 = 1000$.
$\mathtt{exponentof10whichmakes}(100) = 2$, so $\mathbf{10}^{\mathtt{exponentof10whichmakes}(100)} = \mathbf{10}^2 = 100$.
In general, we have our second inverse property:

(2) $\boxed{10^{\mathtt{exponentof10whichmakes}(a)} = a}$

What we are trying to find with $\mathtt{exponentof10whichmakes}(a)$ is exactly that particular real number which makes *a* when we exponentiate the number to the base 10. But 10 raised to the power of that real number will give us *a*.

There is a shorthand for writing the function $\mathtt{exponentof10whichmakes}(x)$, since this is too cumbersome to use in practice. Instead, we name this function the **logarithm to the base 10** (also called the **common logarithm** or just plain "log"), and have the following relationship:

$$\log_{10}(x) = a \text{ means } 10^a = x$$

More shorthand: when the base 10 of the logartithmis omitted, it is understood to mean 10, i.e.,

$$\log(x) = \log_{10}(x).$$

Even more shorthand: parentheses are sometimes omitted, so we could even write:

$$\log x = a \text{ means } 10^a = x$$

Thus, the log of any number ("*x*") is equal to the exponent ("*a*") to which 10 must be raised to get that number.

A few examples of log mathematics are given below.

$$100 = 10^2 \text{ thus } \log 100 = 2$$

$$2.3 \times 10^3 = 10^{3.36} \text{ thus } \log 2.3 \times 10^3 = 3.36$$

$$4.7 \times 10^{-2} = 10^{-1.33} \text{ thus } \log 4.7 \times 10^{-2} = -1.33$$

To write the inverse properties given in Equations (1) and (2) using the log, we would have

(1) $$\log(10^a) = a$$

(2) $$10^{\log(a)} = a$$

To calculate the logarithm of a number, simply enter the number in a calculator and press the "log" key. You can also use a calculator to go in the opposite direction, i.e., take a given number and find out what number it is the log of. For example, find *w* if log(*w*) = −6.78. It may be helpful to remember that whenever we find log(*w*), we are finding an exponent of 10 which gives *w*. So if the logarithm of a number *w* is −6.78, then the number *w* is equal to 10 raised to this power. Or,

$$\log(w) = -6.78, \text{ so } w = 10^{-6.78} = w$$

Another approach to solve for *w* in log (*w*) = −6.78 is to use inverse property (2) after exponentiating both sides to the base 10:

$$\log(w) = -6.78$$
$$10^{\log(w)} = 10^{-6.78}$$
$$w = 10^{-6.78}$$

Some chemistry texts (such as ours) describe the above process as called finding an antilog; however, it is nothing more than the inverse property (2) at work, in "undoing" the log. Thus,

$$\text{antilog}(-6.78) = 10^{-6.78}$$

Many calculators make it easy to take the antilog by providing either a "10^x" key or "inv" key to be used in conjunction with the "log" key. All you have to do is raise 10 to the power of the logarithm. Thus, we can approximate antilog(−6.78) as follows: antilog(−6.78) $= 10^{-6.78} \approx 1.7 \times 10^{-7}$.

Significant Figures in Log and Antilog Calculations

Determining the proper number of significant figures justified in a calculation involving a log or antilog often proves troublesome for students. The following rules should be used when rounding the results of log or antilog calculations.

(1) When calculating the log of a number,
retain as many digits to the right of the decimal point as there are significant figures in the original number.

(2) When calculating the antilog of a number,
retain as many significant figures as there are digits to the right of the decimal point in the original number.

Round the following answers to the appropriate number of significant figures.

(a) $\log 2.000 \times 10^{-5} = -4.698970004$

(b) $\text{antilog } 4.12 = 1.318256739 \times 10^{4}$

Solution

(a) According to rule (1), we retain four digits to the right of the decimal point in the log of the number because there are four significant figures in the original number:

$$\log 2.000 \times 10^{-5} \approx -4.6990$$

(b) According to rule (2), we retain a total of two significant figures in the antilog because there are two digits to the right of the decimal point in the original number.

$$\text{antilog } 4.12 = 10^{4.12} \approx 1.3 \times 10^{4}$$

Computing the Mean and Standard Deviation of a Data Set

If x_i represents an individual measurement in a series, the average or mean ($\bar{x}$) of the series of N measurements is defined as follows:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{N}$$

The average is the sum of the individual measurements divided by the number of measurements (N). The **deviation** (***d_i***) of an individual measurement (x_i) from the mean ($\bar{x}$) can then be calculated as follows,

$$d_i = x_i - \bar{x}$$

A common method for estimating the precision involves calculation of the **standard deviation, *s*.** When N is relatively small, s is calculated according to the following equation:

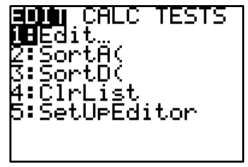
$$s = \sqrt{\frac{d_1^2 + d_2^2 + d_3^2 + d_4^2 + \cdots + d_N^2}{N-1}}$$

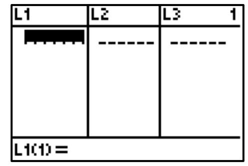
The smaller the value of the standard deviation the greater the precision of the measurement.

Graphing calculators have built-in commands to find the mean and standard deviation. Suppose we have taken the following seven measurements.

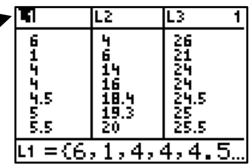
20.06 20.30 20.08 20.04 19.98 20.01 20.07

1. a. Press [STAT] to display the EDIT menu, and select Edit.

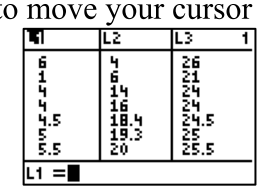




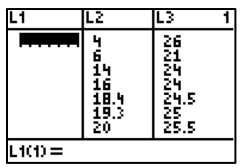
b. If your lists are empty, proceed to part 1c below.
Otherwise, clear the lists as follows:
i. Use [▲] to move your cursor to highlight L_1 at the very top.



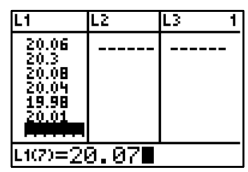
ii. Press [CLEAR]



iii. Press [ENTER].

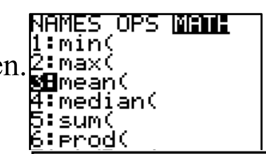


c. Now enter the seven measurements in the calculator into list L_1 .

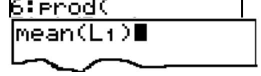


d. Press [2nd] [QUIT] to get to the home screen.
Press [CLEAR] to clear your home screen if needed.

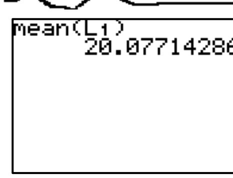
2. a. Press [2nd] [LIST] [►] [►] to get to the MATH menu for lists.
b. Select 3:mean(and press [ENTER] to paste the command on the home screen.



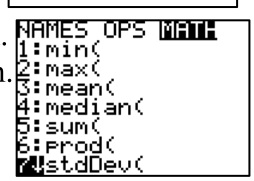
c. Type [2nd] [L_1] and close the parentheses to complete the command:



d. Press [ENTER] to tell the calculator to execute the command:

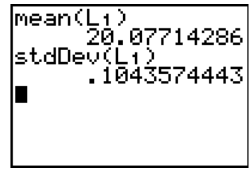


3. a. Press [2nd] [LIST] [►] [►] to get to the MATH menu for lists again.
b. Select 7:stdDev(and press [ENTER] to paste the command on the home screen.



c. Type [2nd] [L_1] and close the parentheses to complete the command:

d. Press ENTER to tell the calculator to execute the command:



Again, the calculator gives us more precision than we can use. Using the rules for **significant figures** we would report the mean as $\bar{x}$ = 20.08 and the standard deviation as s = 0.1044.

Exercises

(1) Complete the following calculations. Pay particular attention to significant figures! Don't use a calculator. (These calculations are not possible on most calculators.)
Report in scientific notation.

(a) $1.39 \times 10^{-107} \times 1.10 \times 10^{103}$ (b) $8.17 \times 10^{105} - 1.20 \times 10^{104}$
(c) $2.6 \times 10^{200} + 5.0 \times 10^{201}$ (d) $3.01 \times 10^{-40} \times 7.0 \times 10^{150}$
(e) $\dfrac{(2.0 \times 10^{23})(1.6 \times 10^{71})}{(4.0 \times 10^{-23})(5.0 \times 10^{11})}$

(2) Find the logarithm of each of the following. Pay particular attention to significant figures!

(a) 1,000. (b) 0.00010 (c) 2.33 (d) 107

(3) Find the anti-log. Pay particular attention to significant figures!

(a) 0.289 (b) 3.289 (c) −2.67 (d) −12.1

(4) A 12 oz. cup of coffee contains about 2.41×10^{18} hydrogen ions. What is the concentration (in moles/liter) of hydrogen ions in a 12 oz. cup of coffee? One liter equals 30.3 oz. One mole of hydrogen ions equals 6.10×10^{23} hydrogen ions. Pay particular attention to significant figures!

(5) The pH of a solution is defined as follows: pH = $-\log[H^+]$, where $[H^+]$ is the concentration of hydrogen ions in moles per liter. Find the pH of the 12 oz. cup of coffee from the previous problem. Pay particular attention to significant figures!

(6) Calculate the mean and the standard deviation for the following set of data.

20.06	19.98
20.30	20.01
20.08	20.02
20.04	20.07

Name_______________________________ Section ____________________

Report Sheet: Powers of Ten

(1) Exponential Notation

(a)____________________

(b)____________________

(c)____________________

(d)____________________

(e)____________________

(2) Logarithms

(a)__________ (b)__________ (c)__________ (d)__________

(3) Antilogs

(a)__________ (b)__________ (c)__________ (d)__________

(4) ____________________

(5) ____________________

(6) Mean and Standard Deviation

$\bar{x}$ = ______________ s = ______________