Engineering Costs and Production Economics, 20 (1990) 81–91
Elsevier

NONLINEAR MULTIPRODUCT CVP ANALYSIS WITH 0–1 MIXED INTEGER PROGRAMMING

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ABSTRACT

This paper presents 0–1 Mixed Integer Programming model for the nonlinear multiproduct Cost–Volume–Profit analysis, which relaxes the assumptions of linear revenue-cost functions and constant fixed cost. In this model, nonlinear revenue and cost functions are approximated by piecewise linear functions, and the joint fixed cost function is represented by a step-increment function. With these features, the required capacity level and the optimal product mix could be determined simultaneously. A hypothetical example, illustrating the model, is presented together with the profit-maximization solution, the breakeven solution, and the target-profit solutions.

1. INTRODUCTION

Cost–Volume–Profit (CVP) analysis is a technique used to analyze the impact on profits of various decisions that affect revenues and costs. One of the limiting assumptions of traditional CVP analysis is that there is only one product or a constant mix of products. In 1961, Jaedicke [1] applied Linear Programming (LP) techniques to construct a CVP model, called a "Product Mix" model in many management accounting or LP texts, which could aid management to determine the optimal product mix, maximizing total profit under some limits (constraints) to production or sales in the case of multiproduct firms. By sensitivity analysis techniques of LP, management could evaluate the impact on profits or product mixes of changing product prices, unit variable costs, resource restrictions, or combinations of these factors [2]. This model could be slightly modified into a linear goal programming model, which could determine the product mix under the breakeven condition or specific profit levels [3].

LP applications have extensively appeared in the accounting literature; for instance, Hartley [4] applies LP techniques to the joint product decision problem, which is another kind of multiproduct CVP analysis problem. However, the LP approach of CVP analysis still retains the following two simplifying assumptions: (1) unit prices and unit variable costs are constant, i.e., the contribution functions are linear within the relevant range, and (2) the fixed cost, which is not divided into specific fixed cost and joint fixed cost, is constant within the relevant range.

To relax these two assumptions, Sheshai et al. [5] utilize Integer Goal Programming (which is 0–1 Mixed Integer Programming strictly speaking) to construct a nonlinear CVP model which expresses each product's contribution as a piecewise linear function and specific fixed cost as a step-increment function. Nevertheless, the model Sheshai et al. present needs the special observance of the association between the contribution function and the varying levels of fixed costs. This special observance becomes unwieldy when there are many kinds of products in the model.

The purpose of this paper is to present a simpler nonlinear CVP model with 0–1 Mixed Integer Programming (0–1 MIP), which relaxes the assumptions mentioned above and considers the discrete capacity extensions. In Section 2, the general form of the nonlinear CVP model is explained in detail; in Section 3, some solution techniques in the literature are discussed; in Section 4, an illustrative example is presented together with the profit-maximization solution, the breakeven solution, and the target-profit solutions.

2. NONLINEAR CVP MODEL

2.1 Assumptions

The nonlinear CVP model presented in this paper is subject to the following assumptions:

(1) In a multiproduct manufacturing firm, total cost is divided into total joint fixed cost, total direct material cost, total direct labor cost, and total other cost.

(2) Joint fixed cost can be measured by machine hours, and is assumed to be a step-increment function, i.e., the acquisition of machine hours comes in indivisible chunks.

(3) The cost functions of total direct material and total direct labor are piecewise linear functions, which approximate the nonlinear cost behaviors, due to quantity discount and higher overtime wage rate.

(4) For each product, "other cost" is the sum of the cost of production and the expenses of administration and marketing except direct material and direct labor costs, and its behavior is a mixed cost pattern.

(5) For each product, its revenue is assumed to be a piecewise linear function which approximates the nonlinear revenue behavior due to the law of diminishing return.

2.2 Presentation of a piecewise linear function

In real life the revenue and cost functions are nonlinear functions, which are approximated by piecewise linear functions in this model. Figure 1 shows an illustration of the piecewise linear function. In Fig. 1, the piecewise linear curve is composed of k successive segments, each with a different slope, indicating different marginal revenue or marginal cost over a different range of volume. Points labeled as $P_1,...,P_k$ are called "bend points" in this paper, whose coordinates are $(V_1,Y_1),..., (V_k, Y_k)$, respectively. Let s_i be the slope of the ith segment; then the formula for calculating Y_i is

$$Y_i = Y_{i-1} + s_i(V_i - V_{i-1}),$$

where $Y_0=0$ and $V_0=0$. Besides, note that V_k is the upper limit of volume. The function value and the associated constraints for this illustration are as follows:

Function value:

$$Y = \pm \sum_{j=1}^{k} Y_j\lambda_j \tag{1.1}$$

Constraints:

$$\lambda_0 - \delta_1 \leqslant 0 \tag{1.2}$$

$$\lambda_j - \delta_j - \delta_{j+1} \leqslant 0 \quad j=1,...,k-1 \tag{1.3}$$

$$\lambda_k - \delta_k \leqslant 0 \tag{1.4}$$

$$\textstyle\sum_{j=0}^{k}\lambda_j = 1 \tag{1.5}$$

$$\textstyle\sum_{j=1}^{k}\delta_j = 1 \tag{1.6}$$

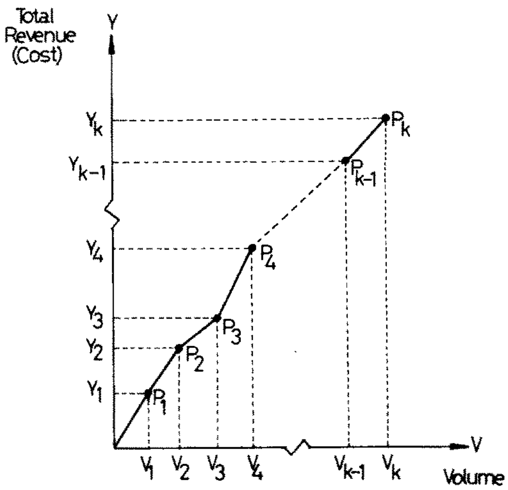

Fig. 1. A piecewise linear function.

$$V-\sum_{j=1}^{k} V_j\lambda_j=0 \tag{1.7}$$

$\delta_j=0,1 \quad j=1,\ldots,k; \quad \lambda_j\geq 0 \quad j=0,\ldots,k$

The term (1.1) will be positive for revenue and negative for cost. $(\delta_1,\delta_2,\ldots,\delta_k)$ is an SOS1 set of 0–1 variables within which exactly one variable must be nonzero; $(\lambda_0,\lambda_1,\ldots,\lambda_k)$ is an SOS2 set of nonnegative variables within which at most two adjacent variables, in the ordering given to the set, can be nonzero [6,7]. By eqn. (1.5), we know that at most two adjacent nonzero λ_j's sum up to 1. Thus, the values of Y and V will be the linear combinations of two adjacent Y_j's and two adjacent V_j's, respectively, by eqns. (1.1) and (1.7). For example, if $\delta_3=1$, then λ_2 and λ_3 sum up to 1, other λ_j's are zero, Y is equal to $Y_2\lambda_2+Y_3\lambda_3$, and V is equal to $V_2\lambda_2+V_3\lambda_3$.

2.3 Description of the model

The complete model this paper presents is as follows:
Maximize total profit function:

$$Z=\sum_{i=1}^{n}\sum_{j=1}^{k_{ir}} R_{ij}\alpha_{ij}-\sum_{j=1}^{k_m} C_{mj}\beta_{mj}-\sum_{j=1}^{k_h} C_{hj}\beta_{hj} -\sum_{i=1}^{n}(F_{oi}\xi_i+C_{oi}X_i)-\sum_{j=0}^{g} F_j\mu_j \tag{2.1}$$

Constraints:

for $i=1,\ldots,n$

$$\alpha_{i0}-\theta_{i1}\leq 0 \tag{2.2}$$

$$\alpha_{ij}-\theta_{ij}-\theta_{i,j+1}\leq 0 \quad j=1,\ldots,k_{ir}-1 \tag{2.3}$$

$$\alpha_{ik_{ir}}-\theta_{ik_{ir}}\leq 0 \tag{2.4}$$

$$\textstyle\sum_{j=0}^{k_{ir}}\alpha_{ij}=1 \tag{2.5}$$

$$\textstyle\sum_{j=1}^{k_{ir}}\theta_{ij}=1 \tag{2.6}$$

$$X_i-\textstyle\sum_{j=1}^{k_{ir}}U_{ij}\alpha_{ij}=0 \tag{2.7}$$

$$X_i-U_{ik_{ir}}\xi_i\leq 0 \tag{2.8}$$

for $s=m,h$

$$\beta_{s0}-\eta_{s1}\leq 0 \tag{2.9}$$

$$\beta_{sj}-\eta_{sj}-\eta_{s,j+1}\leq 0 \quad j=1,\ldots,k_s-1 \tag{2.10}$$

$$\beta_{sk_s}-\eta_{sk_s}\leq 0 \tag{2.11}$$

$$\textstyle\sum_{j=0}^{k_s}\beta_{sj}=1 \tag{2.12}$$

$$\textstyle\sum_{j=1}^{k_s}\eta_{sj}=1 \tag{2.13}$$

$$\textstyle\sum_{i=1}^{n}a_{si}X_i-\sum_{j=1}^{k_s}W_{sj}\beta_{sj}=0 \tag{2.14}$$

$$\textstyle\sum_{i=1}^{n}a_{ti}X_i-\sum_{j=0}^{g}b_j\mu_j\leq 0 \tag{2.15}$$

$$\textstyle\sum_{j=0}^{g}\mu_j=1 \tag{2.16}$$

$\theta_{ij} \;=0,1 \quad i=1,\ldots,n; j=1,\ldots,k_{ir}$
$\xi_i \;=0,1 \quad i=1,\ldots,n$
$\eta_{sj} \;=0,1 \quad s=m,h; j=1,\ldots,k_s$
$\mu_j \;=0,1 \quad j=0,\ldots,g$
other variables are nonnegative.

Equation (2.1) represents the total profit function Z, and eqns. (2.2)–(2.16) are the constraints associated with revenue functions and various cost functions. The discussion which follows explains the elements in the model.

2.3.1 Total revenue

Assume that there are n products and that the revenue function of each product is a piecewise linear function due to price reduction for additional sales. For product i, let the number of units of product i be X_i and let the coordinates of the k_{ir} bend points of its revenue function be (U_{ij}, R_{ij}, $j=1,...,k_{ir}$. Thus, the upper limit of the volume of product i is $U_{ik_{ir}}$.

To express the associated constraints of the piecewise linear revenue function, we introduce some nonnegative variables α_{ij} and some 0–1 variables θ_{ij}, $j=1,...,k_{ir}$; these constraints are represented by eqns. (2.2)–(2.7). Just like the description in Section 2.2, (α_{i0}, $\alpha_{i1},...,\alpha_{ik_{ir}}$) is an SOS2 set, ($\theta_{i1},\theta_{i2},...,\theta_{ik_{ir}}$) is an SOS1 set, the revenue of product i is $\sum_{j=1}^{k_{ir}} R_{ij}\alpha_{ij}$ and the number of units of product i is $\sum_{j=1}^{k_{ir}} U_{ij}\alpha_{ij}$. Because we have n products, there are n sets of eqns. (2.2)–(2.7), and the total revenue is $\sum_{i=1}^{n}\sum_{j=1}^{k_{ir}} R_{ij}\alpha_{ij}$ as shown in the first term of eqn. (2.1).

2.3.2 Total direct material cost

This model assumes that all products utilize the same direct material, whose cost function is a piecewise linear function due to quantity discount. Let the requirement of direct material for one unit of product i be a_{mi} units and the coordinates of the k_m bend points of the total direct material cost function be (W_{mj},C_{mj}),$j=1,...,k_m$. Thus W_{mk_m} is the available volume of direct material.

To express the associated constraints of the piecewise linear function of total direct material cost, we introduce some nonnegative variables β_{mj} and some 0–1 variables η_{mj},$j=1,...,k_m$; these constraints are represented by eqns. (2.9)–(2.14), where the subscript s is replaced by m. Just like the description in Section 2.2., (β_{m1},β_{m2},...,β_{mk_m}) is an SOS2 set, (η_{m1},η_{m2},...,η_{mk_m}) is an SOS1 set, the total direct material cost is $\sum_{j=1}^{k_m} C_{mj}\beta_{mj}$, shown in the second term of eqn. (2.1) and the total requirement of direct material is $\sum_{j=1}^{k_m} W_{mj}\beta_{mj}$, as in eqn. (2.14).

2.3.3 Total direct labor cost

This model assumes that the total direct labor cost function is also a piecewise linear function due to overtime premium. Let the requirement of direct labor for one unit of product i be a_{hi} hours. The representation of the total direct labor cost function, shown in the third term of eqn. (2.1) and eqns. (2.7)–(2.14), is the same as that of the total direct material cost function, except the subscript s is replaced by h. Thus, total direct labor cost is $\sum_{j=1}^{kh} C_{hj}\beta_{hj}$ and total requirement of direct labor is $\sum_{j=1}^{kh} W_{hj}\beta_{hj}$.

2.3.4 Total other cost

For each product, other cost is assumed to be the sum of the cost of production and the expenses of administration and marketing, excluding direct material cost and direct labor cost. Other cost includes the specific fixed cost for each product, so it can be divided into variable and fixed components. This model assumes its behavior is a mixed cost pattern and can be estimated by statistical regression analysis. The representation of total other cost is shown in eqn. (2.8) and the forth term of eqn. (2.1), where F_{oi} stands for the specific fixed cost for product i and C_{oi} is the increment of other cost for producing one more unit of product i. The specific fixed cost F_{oi} can be avoided if we decide not to produce product i. Here, 0–1 variables ξ_i,$i=1,...,n$, are introduced to express this cost phenomenon. When $\xi_i=0$, it means that $X_i=0$ by eqn. (2.8) and that F_{oi} and C_{oi} vanish from the objective function.

2.3.5 Total joint fixed cost

Joint fixed cost is the capacity cost incurred for the common benefit of all products. It usually cannot be specifically identified with a particular product. However, this model assumes that this kind of capacity cost can be measured by the machine hours in a machine-intensive manufacturing company and that its cost function is a step-increment function as shown in Fig. 2. This cost behavior pattern means the capacity extensions in finite jumps, e.g., buying in whole new machines when production is expanded beyond a certain level.

The joint fixed cost is F_0 under the current capacity b_0. If the capacity is successively expanded to $b_1, b_2, \ldots, b_g$, then the joint fixed cost increases to $F_1, F_2, \ldots, F_g$, respectively. Thus, the joint fixed cost function can be expressed by eqns. (2.15), (2.16) and the last term of eqn. (2.1), where a_{ti} is the requirement of machine hours for producing one unit of product i and $(\mu_0, \mu_1, \ldots, \mu_g)$ is an SOS1 set of 0–1 variables. When $\mu_j = 1$, we know that the capacity needs to be expanded to the jth level, i.e., b_j machine hours.

2.4 Remarks

For the model we present, there are some remarks which should be mentioned here. First of all, the piecewise linear function can approximate nonlinear revenue or cost behavior

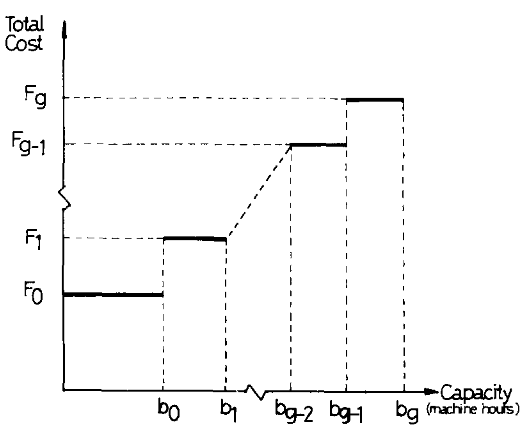

Fig. 2. Total joint fixed cost function.

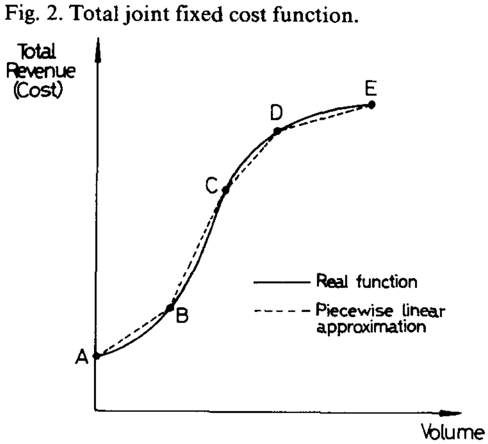

Fig. 3. An illustration of a piecewise linear approximation.

in real world which is estimated either by engineering analysis or by curve-fitting technique. For example, the solid curve in Fig. 3 can be approximated by the dotted lines composed of four segments. If the nonlinear curve is approximated by more segments, this will improve the accuracy of the approximation. It, nevertheless, will increase the numbers of variables and constraints.

Second, one can divide total cost into more items than this model does. Our model has demonstrated the mathematical expressions of various patterns of cost behavior, so the additional cost items can be represented in terms of a linear or nonlinear function according to their behaviors.

Third, product mix solutions are not always in integer units. If an integer solution is required, then we should let X_i be integer and solve the problem by another kind of 0–1 MIP.

Finally, note that the model should be slightly modified in order to acquire the breakeven solution or target-profit solutions. The model described in Section 2.3 is the profit-maximization model, where the objective function is to maximize total profit whose function is shown in eqn. (2.1). To acquire the product mix solution under specific profit level Z_c, the following constraint should be added:

$$Z - e^+ + e^- = Z$$

where Z is the total profit function, and e^+ and e^- are, respectively, positive and negative deviations from Z_c. Thus the objective function is to minimize $e^+ + e^-$. In addition, let Z_c be zero, then the breakeven solution could be acquired.

3. SOLUTION TECHNIQUES

The nonlinear CVP model presented in this paper is a 0–1 MIP model, which involves 0–1 variables and nonnegative variables. There are three 0–1 implicit enumeration algorithms for solving the 0–1 MIP model: the penalty algorithm [8], the partitioning algorithm [9], and the modified partitioning algorithm [10]. A fourth method is the Branch-and-Bound (B&B) algorithm [11]. In fact, the 0–1 implicit enumeration algorithm can be classified as a B&B method. The B&B method has thus far proven to be the most reliable among the various methods for solving the 0–1 MIP model, and all commercial integer programming codes use this method.

The nonlinear CVP model employs special order sets of variables, SOS1 and SOS2 sets, to approximate nonlinear functions by piecewise linear functions. The way in which the B&B algorithm can be modified to deal with SOS1 and SOS2 sets is described in [6, 12]. This kind of approximation will increase the size of the model as well as the computational difficulty. However, Martin and Schrage [13] have developed a subset coefficient reduction method for generating cuts to tighten the 0–1 MIP model prior to implementing an LP based B&B algorithm, in order to reduce the size of the coefficients of the 0–1 variables.

Although the various methods mentioned above can be used to solve the nonlinear CVP model, it will take a long time for practicians to convert the algorithms into computer codes. Fortunately, several ready-made computer codes [14, 15], e.g., the software LINDO [16, 17], are available to solve the nonlinear CVP model. Hence, the practicians could focus their attentions on model building rather than programming.

4. ILLUSTRATION

As an illustration, assume that a manufacturing company has three products, designated as product 1, product 2, and product 3, and that these products utilize the same direct material, the same group of workmen, and the same machines. Next, assume that resource requirement data for one unit of product can be estimated by an engineering analysis method, as shown in Table 1.

We assume, in addition, that revenue and cost data can be estimated by the personnel of marketing and accounting, summarized in Table 2. The model coefficients transformed from the estimated data are also included in Table 2. Among these data, the functions of each product's revenue, direct material cost, and direct labor cost are piecewise linear functions similar to the function shown in Fig. 1; the joint fixed cost function is a step-increment function similar to the function shown in Fig. 2; the other cost function of each product is a mixed cost pattern.

TABLE 1

Resource requirement data for one unit of product

Product i	Resource requirement		
	material a_{mi}	labor a_{hi}	machine a_{ti}
1	3.7	4	8
2	3.6	2	6
3	4.0	3	6

In this section, we will explain how to formulate the model for this illustrative case by 0–1 MIP and present its profit-maximization solution and target-profit solutions.

4.1 Model formulation

4.1.1 Total revenue

From Table 2(a), (b) and (c) we know that the revenue functions of all three products are piecewise linear functions with two segments (i.e., $k_{1r}=k_{2r}=k_{3r}=2$). For product 1, the revenue and the associated constraints are:
Revenue:

$$21\,600\,\alpha_{11}+34\,400\,\alpha_{12}$$

Constraints:

$$\alpha_{10}-\theta_{11}\leqslant 0$$

$$\alpha_{11}-\theta_{11}-\theta_{12}\leqslant 0$$

$$\alpha_{12}-\theta_{12}\leqslant 0$$

TABLE 2

Revenue and cost data of illustrative case

(a) *Revenue data of product 1 (i=1)*

j	volume	marginal revenue ($)	model coeff. U_{1j}	R_{1j} ($)
1	1- 600	36	600	21 600
2	601-1000	32	1000	34 400

(b) *Revenue data of product 2 (i=2)*

j	volume	marginal revenue ($)	model coeff. U_{2j}	R_{2j} ($)
1	1-600	28	600	16 800
2	601-900	25	900	24 300

(c) *Revenue data of product 3 (i=3)*

j	volume	marginal revenue ($)	model coeff. U_{3j}	R_{3j} ($)
1	1-500	30	500	15 000
2	501-800	26	800	23 400

(d) *Direct material cost data*

j	volume	marginal cost ($)	model coeff. W_{mj}	C_{mj} ($)
1	1- 5 000	1.0	5 000	5000
2	5001- 8 000	0.8	8 000	7400
3	8001-10 000	0.7	10 000	8800

(e) *Direct Labor cost data*

j	volume	marginal cost ($)	model coeff. W_{hj}	C_{hj} ($)
1	1-4000	2	4000	8 000
2	4001-6000	3	6000	14 000

(f) *Other cost data*

Product i	components: fixed F_{oi} ($)	variable C_{oi} ($)
1	1800	6
2	2100	5
3	2200	4

(g) *Joint fixed cost data*

j	increment of machine hours	increment of joint fixed cost ($)	model coeff. b_j	F_j ($)
0	8000	8000	8 000	8 000
1	2000	2000	10 000	10 000
2	4000	4000	12 000	12 000

$$\alpha_{10}+\alpha_{11}+\alpha_{12}=1$$

$$\theta_{11}+\theta_{12}=1$$

$$X_1-600\,\alpha_{11}-1000\,\alpha_{12}=0.$$

Next, the analogous expressions for product 2 are:

Revenue:

$$16\,800\,\alpha_{21}+24\,300\,\alpha_{22}$$

Constraints:

$$\alpha_{20}-\theta_{21}\leqslant 0$$

$$\alpha_{21}-\theta_{21}-\theta_{22}\leqslant 0$$

$$\alpha_{22}-\theta_{22}\leqslant 0$$

$$\alpha_{20}+\alpha_{21}+\alpha_{22}=1$$

$$\theta_{21}+\theta_{22}=1$$

$$X_2-600\,\alpha_{21}-900\,\alpha_{22}=0.$$

Finally, the analogous expressions for product 3 are:

Revenue:

$$15\,000\,\alpha_{31}+23\,400\,\alpha_{32}$$

Constraints:

$$\alpha_{30}-\theta_{31}\leqslant 0$$

$$\alpha_{31}-\theta_{31}-\theta_{32}\leqslant 0$$

$$\alpha_{32}-\theta_{32}\leqslant 0$$

$$\alpha_{30}+\alpha_{31}+\alpha_{32}=1$$

$$\theta_{31}+\theta_{32}=1$$

$$X_3-500\,\alpha_{31}-800\,\alpha_{32}=0.$$

4.1.2 Total direct material cost

From Table 2(d), we know that the direct material cost function is a piecewise linear function with three segments (i.e., $k_m=3$). It is because of quantity discount; the marginal cost of material is $1 under 5000 units, $0.8 between 5001 and 8000 units, and $0.7 between 8001 and 10 000 units. Furthermore, the

material requirements to produce one unit of product 1, product 2 and product 3 are 3.7, 3.6 and 4 units, respectively. Therefore, total direct material cost and the associated constraints are:
Total direct material cost:

$$5000\,\beta_{m1}+7400\,\beta_{m2}+8800\,\beta_{m3}$$

Constraints:

$$\beta_{m0}-\eta_{m1}\leqslant 0$$

$$\beta_{m1}-\eta_{m1}-\eta_{m2}\leqslant 0$$

$$\beta_{m2}-\eta_{m2}-\eta_{m3}\leqslant 0$$

$$\beta_{m3}-\eta_{m3}\leqslant 0$$

$$\beta_{m0}+\beta_{m1}+\beta_{m2}+\beta_{m3}=1$$

$$\eta_{m1}+\eta_{m2}+\eta_{m3}=1$$

$$3.7X_1+3.6X_2+4X_3-5000\,\beta_{m1}-8000\,\beta_{m2}-10\,000\,\beta_{m3}=0.$$

4.1.3 Total direct labor cost

From Table 2(e), we know that the direct labor cost function is a piecewise linear function with two segments (i.e., $k_h=2$). Here, we assume that this company executes the time-work wage system; the available normal labor hours are 4000 hours with the hourly rate of \$2, and the available overtime labor hours are 2000 hours with the hourly rate of \$3. Moreover, the labor requirements to produce one unit of product 1, product 2 and product 3 are 4, 3 and 3 hours, respectively. Thus, total direct labor cost and the associated constraints are:
Total direct labor cost:

$$8000\beta_{h1}+14\,000\,\beta_{h2}$$

Constraints:

$$\beta_{h0}-\eta_{h1}\leqslant 0$$

$$\beta_{h1}-\eta_{h1}-\eta_{h2}\leqslant 0$$

$$\beta_{h2}-\eta_{h2}\leqslant 0$$

$$\beta_{h0}+\beta_{h1}+\beta_{h2}=1$$

$$\eta_{h1}+\eta_{h2}=1$$

$$4X_1+2X_2+3X_3-4000\,\beta_{h1}-6000\,\beta_{h2}=0.$$

4.1.4 Total other cost

Other cost data of the three products are shown in Table 2(f). Total other cost and the associated constraints are:
Total other cost:

$$1800\,\xi_1+6X_1+2100\,\xi_2+5X_2+2200\,\xi_3+4X_3$$

Constraints:

$$X_1-1000\,\xi_1\leqslant 0$$

$$X_2-900\,\xi_2\leqslant 0$$

$$X_3-800\,\xi_3\leqslant 0.$$

4.1.5 Joint fixed cost

From Table 2(g), we know that the current capacity is 8000 machine hours and that it can be expanded to 10 000 or 12 000 machine hours with the additional cost of \$2000 or \$4000. Besides, the machine hours requirements to produce one unit of product 1, product 2 and product 3 are 8, 6 and 6 hours, respectively. Thus, total joint fixed cost and the associated constraints are:
Total joint fixed cost:

$$8000\,\mu_0+10\,000\,\mu_1+12\,000\,\mu_2$$

Constraints:

$$8X_1+6X_2+6X_3-8000\,\mu_0-10\,000\,\mu_1-12\,000\,\mu_2\leqslant 0$$

$$\mu_0+\mu_1+\mu_2=1.$$

4.2 Solution

4.2.1 Profit-maximization solution

If the goal of management is to achieve the maximum profit, then the objective function is to maximize the following function:

$$Z=(\text{total revenue})-(\text{total direct material cost})$$

− (total direct labor cost)
− (total other cost) − (total joint fixed cost)
$= (21\,600\,\alpha_{11} + 34\,400\,\alpha_{12} + 16\,800\,\alpha_{21} + 24\,300\,\alpha_{22} + 15\,000\,\alpha_{31} + 23\,400\,\alpha_{32})$
$- (5000\,\beta_{m1} + 7400\,\beta_{m2} + 8800\,\beta_{m3})$
$- (8000\,\beta_{h1} + 14\,000\,\beta_{h2})$
$- (1800\,\xi_1 + 6X_1 + 2100\,\xi_2 + 5X_2 + 2200\,\xi_3 + 4X_3)$
$- (8000\,\mu_0 + 10\,000\,\mu_1 + 12\,000\,\mu_2)$

We solve this 0–1 MIP problem by software LINDO and obtain the following optimal solution:

$X_1=450,\quad \xi_1=1,\quad \theta_{11}=1,\quad \theta_{12}=0,$
$X_2=600,\quad \xi_2=1,\quad \theta_{21}=0,\quad \theta_{22}=1,$
$X_3=800,\quad \xi_3=1,\quad \theta_{31}=0,\quad \theta_{32}=1,$
$\alpha_{10}=0.25,\quad \alpha_{11}=0.75,\quad \alpha_{12}=0,\quad \mu_0=0,$
$\alpha_{20}=0,\quad \alpha_{21}=1,\quad \alpha_{22}=0,\quad \mu_1=0,$
$\alpha_{30}=0,\quad \alpha_{31}=0,\quad \alpha_{32}=1,\quad \mu_2=1,$
$\beta_{m0}=0, \beta_{m1}=0.325, \beta_{m2}=0.675, \beta_{m3}=0, \eta_{m1}=0, \eta_{m2}=1,$
$\eta_{m3}=0, \beta_{h0}=0, \beta_{h1}=0.3, \beta_{h2}=0.7, \eta_{h1}=0, \eta_{h2}=1.$

Accordingly, the optimal product mix is $(X_1,X_2,X_3=(450,600,800)$, which requires 7025 $\times(450\cdot 3.7+600\cdot 3.6+800\cdot 4)$ units of material, $6000\times(450\cdot 4+600\cdot 3+800\cdot 3)$ labor hours, and $12\,000\times(450\cdot 8+600\cdot 6+800\cdot 6)$ machine hours. Total revenue, total direct material cost, total direct labor cost, total other cost, and total joint fixed cost are \$56 400, \$6620, \$12 200, \$15 000, and \$12 000 respectively, calculated by the expressions mentioned above. Thus, total profit is \$10 580.

4.2.2 Target-profit solutions

As mentioned previously, the following constraint should be added in order to acquire the product mix solution under specific profit level Z_c:

$$Z - e^+ + e^- = Z_c,$$

and the objective function is changed into minimizing $e^+ + e^-$. The product mix solutions under various target profit levels for the illustrative case are shown in Table 3. The breakeven product mix (when $Z_c=0$) is $(X_1,X_2,X_3) = (535.5192,0,0)$; it means that only product 1 is produced. From Table 5, we see that the achievable maximum profit is \$10 580, above which the negative deviation variable e^-, indicating the amount of not achieving the target profit, will be positive.

TABLE 3

Target-profit solutions

Target profit Z_c (\$)	Product mix (in units)			deviations	
	X_1	X_2	X_3	e^+	e^-
0	535.5192	0	0	0	0
1 000	262.2951	0	800	0	0
2 000	316.9399	0	800	0	0
3 000	371.5847	0	800	0	0
4 000	582.5137	600	0	0	0
5 000	545.4546	0	745.4545	0	0
6 000	156.2842	600	500	0	0
7 000	0	600	785.6757	0	0
8 000	0	660.0610	800	0	0
9 000	0	736.2805	800	0	0
10 000	787.3187	900	0	0	0
10 580	450	600	800	0	0
11 000	450	600	800	0	420
12 000	450	600	800	0	1420

5. DISCUSSION

In order to approximate the nonlinear revenue and cost functions by the piecewise linear functions, the nonlinear CVP model involves a rather larger number of variables and constraints as shown in Table 4. For the illustrative example in Section 4, $n=3$, $k_{1r}=k_{2r}=k_{3r}=2$, $k_m=3$, $k_h=2$, and $g=2$; therefore, there are 36 constraints, 19 nonnegative variables, and 17 0–1 variables.

Accordingly, the nonlinear CVP model will become larger, when used to describe the real world problems. Suppose that all the nonlinear revenue and cost functions can be approximated by a 5-segment piecewise linear function (i.e., $k_{ir}=k_m=k_h=5$) and that $g=5$. Under this condition, if there are 10 products ($n=10$), then the model will contain 120 constraints, 82 nonnegative variables, and 76 0–1

TABLE 4

Size of the nonlinear CVP model

		if $k_{ir}=k_m=k_h=k$
Constraints	$\sum_{i=1}^{n} k_{ir}+k_m+k_h+5n+10$	$(n+2)k+5n+10$
Nonnegative variables	$\sum_{i=1}^{n} k_{ir}+k_m+k_h+2n+2$	$(n+2)k+2n+2$
0–1 variables	$\sum_{i=1}^{n} k_{ir}+k_m+k_h+n+g+1$	$(n+2)k+n+g+1$
Nonzero coeff. & RHS	$7(\sum_{i=1}^{n} k_{ir}+k_m+k_h)+12n+3g+12$	$7(n+2)k+12n+3g+12$

Note: k_{ir} = number of segments of the piecewise linear function for the ith product's revenue,
k_m = number of segments of the piecewise linear function for the total direct material cost,
k_h = number of segments of the piecewise linear function for total direct labor cost,
g = number of ranges of the step-increment function for total joint fixed cost,
n = number of products.

TABLE 5

Maximum size of input matrix on LINDO*

Input	Super LINDO/PC for IBM PC	LINDO for IBM mainframe
Row (obj. fn. plus constraints)	250	4 999
Columns (variables)**	500	14 999
0–1 variables	490	14 900
Nonzeros	8000	60 000

*These data are obtained by the HELP function of the software LINDO.
**The variables include nonnegative variables and 0–1 variables.

variables; if there are 100 products ($n=100$), then the model will contain 1020 constraints, 712 nonnegative variables, and 616 0–1 variables.

What we have to be concerned with is the number of 0–1 variables of the model, which is one of the primary determinants of computational difficulty for solving a 0–1 MIP model. The nonlinear CVP model contains so many 0–1 variables that it seems that we can not apply the model to the real world situations. However, the rapid developments of computer hardware and mathematical programming software have made the application possible. For example, the software LINDO, whose maximum size of input matrix is shown in Table 5, can handle at most 22 products on the IBM PC-version and 497 products on the IBM mainframe-version for our model under the condition $k_{ir}=k_m=k_h=g=5$.

6. CONCLUSION

Two of primary assumptions for traditional CVP analysis are the absence of step costs and the requirement of linear revenue and cost functions, which are based on the relevant range concept. Under the relevant range concept, the average product-mix model assumes that the relevant range, within which the firm plans to operate, is known in advance and then the optimal product mix is determined by Linear Programming. With the features of the model we present, including piecewise linear revenue and cost functions and step-increment joint fixed cost function, we can determine the required capacity level and the optimal product mix simultaneously. Thus, we need not to restrict decisions to the specific relevant range if we formulate the revenue and cost functions with the possible range of activities. Although the nonlinear CVP model presented in this paper contains a rather large number of variables and constraints, the rapid developments of computer hardware and mathematical programming software have overcome this. In conclusion, it seems possible that the nonlinear CVP analysis will be adopted as extensively as the traditional CVP analysis was.

ACKNOWLEDGEMENT

This research was supported by the National Science Council of the Republic of China under grant NCS78-0415-E011-03.

REFERENCES

1 Jaedicke, R.K., 1961. Improving breakeven analysis by linear programming techniques. NAA Bulletin, March: 5–12.

2 Killough, L.N. and Leininger, W.E., 1984. Cost Accounting: Concepts and Techniques for Management. West Publishing Company, St. Paul, MN, pp. 398–405.

3 Charnes, A., Cooper, W.W. and Ijiri, Y., 1963. Breakeven budgeting and programming to goals. J. Accounting Res., 1(1): 16–43.

4 Hartley, R.V., 1971. Decision making when joint products are involved. The Accounting Rev., XLVI(October): 746–755.

5 Sheshai, K.M. El., Harwood, G.B. and Hermanson, R.H., 1977. Cost volume profit analysis with integer goal programming. Manage. Accounting, 54(4): 43–47.

6 Beale, E.M.L. and Tomlin, J.A., 1970. Special facilities in a general mathematical programming system for nonconvex problems using ordered sets of variables. In: Lawrence, J., (Ed.), Proc. 5th Int. Conf. Oper. Res., Tavistock, London, 447–454.

7 Williams, H.P., 1985. Model Building in Mathematical Programming, 2nd edn. John Wiley & Sons Ltd., New York, pp. 173–177.

8 Driebeek, N.J., 1966. An algorithm for the solution of mixed integer programming problems. Manage. Sci., 12(7): 576–587.

9 Benders, J.F., 1962. Partitioning procedures for solving mixed-variables programming problems. Numer. Math., 4: 239–252.

10 Lemke, C.E. and Spielberg, K., 1967. Direct search zero-one and mixed integer programming. Oper. Res., 15: 892–914.

11 Davis, R.E., Kendrick, D.A. and Weitzman, M., 1971. A branch-and-bound algorithm for 0–1 mixed integer programming problems. Oper. Res., 19: 1036–1044.

12 Beale, E.M.L. and Forrest, J.J.H., 1976. Global optimization using special ordered sets. Math. Programming, 10: 52–69.

13 Martin, R.K. and Schrage, L., 1985. Subset coefficient reduction cuts for 0–1 mixed-integer programming. Oper. Res., 33(3): 505–526.

14 Johnston, E.L. and Powell, S., 1978. Integer programming codes. In: Greenberg, H., (Ed.), Design and Implementation of Optimization Software. Sijthoff & Noordhoff, Alphen aan den Rijn.

15 Land, A.H. and Powell, S., 1981. A survey of available computer codes to solve integer linear programming problems. QSSMS Discussion Paper No. 59, Univ. Kent, Canterbury (April).

16 Schrage, L., 1986. Linear, Integer and Quadratic Programming with LINDO, 3rd edn. The Scientific Press, Redwood City, CA.

17 Schrage, L., 1987. Linear, Integer, and Quadratic Programming with LINDO-User's Manual, 3rd edn. The Scientific Press, Redwood City, CA.

(Received January 31, 1989; accepted December 21, 1989)