Classification of Rational Holomorphic Maps from $\mathbb{B}^2$ into $\mathbb{B}^N$ with Degree 2

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March 1, 2009

1 Introduction

Denote by $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ the space of all rational proper holomorphic maps from the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$ into the unit ball $\mathbb{B}^N \subset \mathbb{C}^N$. We recall that $F$ and $G \in \text{Prop}(\mathbb{B}^n, \mathbb{B}^N)$ are said to be equivalent if there are automorphisms $\sigma \in \text{Aut}(\mathbb{B}^n)$ and $\tau \in \text{Aut}(\mathbb{B}^N)$ such that $F = \tau \circ G \circ \sigma$.

In this paper, we study the classification problem for elements in $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ with degree two. For an element $F$ in $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$, there is a naturally associated invariant $Rk_F \leq 1$, called the geometric rank of the map. Since $F$ is linear if and only if its geometric rank (for the definition, see §2) $Rk_F = 0$, we only need to consider maps with geometric rank $Rk_F = 1$. By using Cayley transformation $\rho_k : \mathbb{H}^k \to \mathbb{B}^k$ where $\mathbb{H}^k$ is the Siegel upper-half space (see §2), studying $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ is equivalent to studying $\text{Rat}(\mathbb{H}^2, \mathbb{H}^N)$.

Making use of results obtained in the previous work [HJX06] [CJX06], we give a complete description for the modular space for maps in $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ with degree $\leq 2$ under the above mentioned equivalence relation. Our main result is the following Theorem 1.1.

Theorem 1.1 (i) Any nonlinear map in $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2 is equivalent to a map $(F, 0)$ where $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^5)$ is of one of the following forms:

(I): $F = (G_t, 0)$ where $G_t \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$G_t(z, w) = (z^2, \sqrt{1 + \cos^2 t \, zw}, (\cos t)w^2, (\sin t)w), \quad 0 \leq t < \pi/2.$$  \hspace{1cm} (1)

(IIA): $F = (F_\theta, 0)$ where $F_\theta \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$F_\theta(z, w) = (z, (\cos \theta)w, (\sin \theta)zw, (\sin \theta)w^2), \quad 0 < \theta \leq \frac{\pi}{2}.$$  \hspace{1cm} (2)
(IIC): \( F = F_{c_1, c_3, e_1, e_2} = \rho^1_5 \circ F \circ \rho_2 = (f, \phi_1, \phi_2, \phi_3, g) \in \text{Rat}(\mathbb{H}^2, \mathbb{H}^5) \) is of the form:

\[
\begin{align*}
  f &= \frac{z + (\frac{1}{2} + ie_1)zw}{1 + ie_1w + e_2w^2}, \\
  \phi_1 &= \frac{z^2}{1 + ie_1w + e_2w^2}, \\
  \phi_2 &= \frac{c_1zw}{1 + ie_1w + e_2w^2}, \\
  \phi_3 &= \frac{c_3w^2}{1 + ie_1w + e_2w^2}, \\
  g &= \frac{w + ie_1w^2}{1 + ie_1w + e_2w^2},
\end{align*}
\]

where \( c_1, c_3 > 0, -e_1, -e_2 \geq 0, e_1e_2 = c_3^2, -e_1 - e_2 = \frac{1}{4} + c_3^2 \), satisfying one of the following conditions: either

\[
\begin{cases}
  e_1 = -\left(\frac{1}{4} + c_3^2\right) - \sqrt{(\frac{1}{4} + c_3^2)^2 - 4c_3^2}, \\
  0 < 4c_3^2 \leq (\frac{1}{4} + c_3^2)^2,
\end{cases}
\]

or

\[
\begin{cases}
  e_1 = -\left(\frac{1}{4} + c_3^2\right) + \sqrt{(\frac{1}{4} + c_3^2)^2 - 4c_3^2}, \\
  \frac{1}{2}c_1^2 + c_4^2 \leq 4c_3^2 \leq (\frac{1}{4} + c_3^2)^2.
\end{cases}
\]

(ii) Any two maps in \( \text{Rat}(\mathbb{H}^2, \mathbb{H}^5) \) in the form of types (I), (IIA), and (IIC) above are equivalent if and only if they are identical.

Next, we give a review on the development of this problem and outline the proof for Theorem 1.1 as follows. For some notations to be used, we refer the reader to §2.

• A result obtained in [HJX06] A classification result was proved in the last section of [HJX06] under the action of the isotropic automorphism groups of the Heisenberg hypersurfaces, which gives in particular the following: Any map \( F \) in \( \text{Rat}(\mathbb{H}^2, \mathbb{H}^N) \) with \( \text{deg}(F) = 2 \) is equivalent to a map \( (G, 0) \) where \( G = (f, \phi_1, \phi_2, \phi_3, g) \in \text{Rat}(\mathbb{H}^2, \mathbb{H}^5) \) is of the form (see also Lemma 2.3 below)

\[
\begin{align*}
  f(z, w) &= \frac{z - 2bw^2 + (\frac{1}{2} + ie_1)zw}{1 + ie_1w + e_2w^2 - 2bw}, \\
  \phi_1(z, w) &= \frac{z^2 + zbw}{1 + ie_1w + e_2w^2 - 2bw}, \\
  \phi_2(z, w) &= \frac{c_1zw}{1 + ie_1w + e_2w^2 - 2bw}, \\
  \phi_3(z, w) &= \frac{c_3w^2}{1 + ie_1w + e_2w^2 - 2bw}, \\
  g(z, w) &= \frac{c_2w^2 + c_1zw}{1 + ie_1w + e_2w^2 - 2bw},
\end{align*}
\]

where \( b, -e_1, -e_2, c_1, c_2, c_3 \) are real non-negative numbers satisfying \( e_1e_2 = c_3^2, -e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2, -be_2 = c_1c_2, \) and \( c_3 = 0 \) if \( c_1 = 0 \).

Since \( b \) and \( e_2 \) are determined by \( c_1, c_3, e_1 \) and \( e_2 \), a map in the form of (5) is determined by \( c_1, c_3, e_1 \) and \( e_2 \). We denote a map of the form (5) determined by \( c_1, c_3, e_1 \) and \( e_2 \) to be

\[
F_{(c_1, c_3, e_1, e_2)} \in \mathcal{K}.
\]

(6)
Sometimes we regard a such map \(F_{(c_1,c_3,e_1,e_2)}\) as a point: \((c_1,c_3,e_1,e_2) \in \mathcal{K}\). It was unclear in [HJX06] which of the coefficients \(e_1, e_2, c_1\) and \(c_3\) of \(F\) are independent parameters.

**Review of the result in [CJX06]** In [CJX06], by obtaining an extra equation, we got a clearer picture on the maps in (5).

For any \(F \in \text{Rat}(\mathbb{H}^2,\mathbb{H}^5)\) with \(\text{deg}(F) = 2\), if the geometric rank of \(F\) at the origin is one: \(Rk_F(0) = 1\), then by a normalization procedure (see Lemma 2.2 and 2.3 below, or [Hu 03][HJX06]), \(F\) is equivalent to another map \(F^{***} \in \text{Rat}(\mathbb{H}^2,\mathbb{H}^5)\) of the form (5). Also we can associate a family of maps \(F_p \in \text{Rat}(\mathbb{H}^2,\mathbb{H}^5)\) for any \(p \in \partial \mathbb{H}^2\) (see § 2 below). Let us define \(\Xi_F := \{p \in \partial \mathbb{H}^2 \mid Rk_{F_p}(0) = 0\}\) to be the set of \(p\) at which the geometric rank of \(F_p\) at the origin is zero. If \(p \notin \Xi_F\), we obtain a normalized map \((F_p)^{***}\) that is of the form (5), and we define a real analytic function \(W(F_p^{***}) = c_1(p)^2 - e_1(p) - e_2(p)\) where \(c_1(p), e_1(p)\) and \(e_2(p)\) are the coefficients of \(F^{***}\) as in (5).

The desired extra equation is obtained by moving up \(p\) to the extremal value as follows. We choose a sequence of \(p_m \in \partial \mathbb{H}^2 - \Xi_F\) such that \(Rk_{F_{p_m}}(0) = 1, p_m \to p_0 \in \partial \mathbb{H}^2\) and \(\lim_m W(F_{p_m}^{***}) = \inf_{p \in \partial \mathbb{H}^2 - \Xi_F} W(F_p^{***})\). The minimum property implies the vanishing of derivatives of the function \(W(F_p^{***})\) at \(p_0\), which derives the extra equation.

If \(p_0 = \infty\), by [CJX06, § 4] we can similarly write

\[
F_{p_m}^{***} = (F_{p_0})^{***}_{q_m}
\]  

(7)

where \(q_m \in \partial \mathbb{H}^2\) and \(q_m \to 0\). Then it implies by [CJX06, Lemma 2.5] that \(Rk_{F_{p_0}}(0) = 1\), and that \(F\) is equivalent to \(F^{***}\) which is of the form (5) and with the minimum property \(W(F_{p_0}^{***}) = \inf_{p \in \partial \mathbb{H}^2 - \Xi_F} W(F_p^{***})\). The minimum property implies the vanishing of derivatives of the function \(W(F_p^{***})\) at \(p_0\), which derives the extra equation.

If \(p_0 = \infty\), by [CJX06, § 4] we can similarly write

\[
F_{p_m}^{***} = (\tau_\infty \circ F \circ \sigma_\infty)^{***}_{q_m}
\]  

(8)

where \(\sigma_\infty \in \text{Aut}(\partial \mathbb{H}^2), \tau_\infty \in \text{Aut}(\partial \mathbb{H}^5), q_m \in \partial \mathbb{H}^2\) and \(q_m \to 0\) so that, by the same argument above, \(Rk_{\tau_\infty \circ F \circ \sigma_\infty}(0) = 1\) and that \(F\) is equivalent to \((\tau_\infty \circ F \circ \sigma_\infty)^{***}\) which is of the form (5). The minimum property also derives the extra equation.

With the extra equation described above, it was proved in [CJX06] that \(F\) is equivalent to another map \(F_{c_1,c_3,e_1,e_2} \in \mathcal{K}\) satisfying the property

\[
W((F_{c_1,c_3,e_1,e_2})_p) \geq W((F_{c_1,c_3,e_1,e_2})_0), \quad \forall p \in \partial \mathbb{H}^2 \text{ near } 0.
\]  

(9)

and that the new map \(F_{c_1,c_3,e_1,e_2}\) is of the form in one of the following types:

1. If \(F_{0,0,e_1,e_2} = (f, \phi_1, \phi_2, \phi_3, g)\) is of the form

\[
\begin{align*}
f &= \frac{z + (\frac{1}{2} + i e_1)w}{1 + i e_1 w + e_2 w^2}, \\
\phi_1 &= \frac{z^2}{1 + i e_1 w + e_2 w^2}, \\
\phi_2 &= \frac{e_2 w^2}{1 + i e_1 w + e_2 w^2}, \\
\phi_3 &= 0, \\
g &= \frac{w + i e_1 w^2}{1 + i e_1 w + e_2 w^2}
\end{align*}
\]  

(10)
where \( e_1e_2 = c_2^2 \) and \(-e_1 - e_2 = 1\). Here \( e_2 \in [-\frac{1}{4}, 0) \) is a parameter. It then corresponds to the family \( \{ \Gamma \} \) \( \subseteq t < \pi/2 \) in (I). When \( e_2 = -\frac{1}{4} \), \( F_{0,0,e_1,e_2} \) corresponds to \( \Gamma \), i.e., \((z, w) \mapsto (z^2, \sqrt{2zw}, w^2, 0)\); when \( e_2 \to 0 \), \( F_{0,0,e_1,e_2} \) goes to \( \Gamma_{\pi/2} = F_{\pi/2} \), i.e., \((z, w) \mapsto (z, zw, w^2)\).

(I) \( F_{c_1,0,e_1,0} = (f, \phi_1, \phi_2, \phi_3, g) \) is of the form

\[
f = \frac{z + \left(\frac{i}{2} + i e_1\right)zw}{1 + i e_1 w}, \quad \phi_1 = \frac{z^2}{1 + i e_1 w}, \quad \phi_2 = \frac{c_1 zw}{1 + i e_1 w}, \quad \phi_3 = 0, \quad g = w
\]  

where \(-e_1 = \frac{1}{4} + c_1^2\) and \( c_1 \in (0, \infty) \) is a parameter. It corresponds to the family \( \{ \Gamma \} \) \( \subseteq t < \pi/2 \) in (2). When \( c_1 = 0 \), \( F_{c_1,0,e_1,0} \) corresponds to \( \Gamma_{\pi/2} \); when \( c_1 \to \infty \), \( F_{c_1,0,e_1,0} \) goes to the linear map, i.e., \((z, w) \mapsto (z, w, 0)\).

(IIB) \( F_{c_1,0,e_2} = (f, \phi_1, \phi_2, \phi_3, g) \) is of the form:

\[
f = \frac{z + \frac{i}{2} zw}{1 + e_2 w^2}, \quad \phi_1 = \frac{z^2}{1 + e_2 w^2}, \quad \phi_2 = \frac{c_1 zw}{1 + e_2 w^2}, \quad \phi_3 = 0, \quad g = \frac{w}{1 + e_2 w^2},
\]

where \(-e_2 = \frac{1}{4} + c_1^2\) and \( c_1 \in (0, \infty) \) is a parameter. Notice that when \( c_1 \to 0 \), the map \( F_{c_1,0,e_2} \) goes to the map \( \Gamma \), i.e. the one in type (I) when \( e_2 = -\frac{1}{4}\).

(IIC) \( F_{c_1,c_2,e_1,e_2} = (f, \phi_1, \phi_2, \phi_3, g) \) is of the form:

\[
f = \frac{z + (\frac{i}{2} + i e_1)zw}{1 + i e_1 w + e_2 w^2}, \quad \phi_1 = \frac{z^2}{1 + i e_1 w + e_2 w^2}, \quad \phi_2 = \frac{c_1 zw}{1 + i e_1 w + e_2 w^2}, \quad \phi_3 = \frac{w + i e_1 w^2}{1 + i e_1 w + e_2 w^2}, \quad g = \frac{w + i e_1 w^2}{1 + i e_1 w + e_2 w^2},
\]

where \( c_1, c_2 > 0, -e_1, -e_2 \geq 0, \quad e_1 e_2 = c_2^2, \quad -e_1 - e_2 = \frac{1}{2} + c_1^2.\)

For any map \( F_{c_1,c_2,e_1,e_2} \) in one of these four types, we denote \( F_{c_1,c_2,e_1,e_2} \), or \((c_1, c_2, e_1, e_2)\), \( \in \mathcal{K}_{I}, \mathcal{K}_{IIA}, \mathcal{K}_{IIB}, \) and \( \mathcal{K}_{IIC} \), respectively.

Recall from (33) [CJX06]

\[
F \text{ can be embedded into } \mathbb{H}^4 \leftrightarrow c_3 = 0.
\]  

Concerning the proof of Theorem 1.1, our main idea to establish following formula (see (33)):

\[
\mathcal{W}(F_{\Gamma(\theta + \Delta t)}) = \mathcal{W}(F_{\Gamma(\theta)}) + [4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(\theta))\Im(q_1(t))\Delta t + o(|\Delta t|).
\]  

One crucial point is that the term \([4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(\theta))\) is always non-negative so that it allows us to reduce the study of (9) into the study for \( \Im(q_1(t)) \).

We’ll prove in Lemma 3.4 below that indeed

\[
there \ is \ no \ map \ F \ satisfying \ both \ (9) \ and \ (12),
\]  

4
and that a map 

\[ F \text{ satisfies (9) and (13)} \iff F \text{ satisfies (13), (3) and (4),} \]

which proves Theorem 1.1(i). To prove Theorem 1.1(ii), we first prove its local version (see Corollary 4.3). Then we shall find a way to reduce the global problem into the local one.

2 Notation and preliminaries

- **Maps between balls** Write \( \mathbb{H}^n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im}(w) > |z|^2\} \) for the Siegel upper-half space. Similarly, we can define the space \( \text{Rat}(\mathbb{H}^n, \mathbb{H}^N), \text{Prop}_k(\mathbb{H}^n, \mathbb{H}^N) \) and \( \text{Prop}(\mathbb{H}^n, \mathbb{H}^N) \) respectively. Since the Cayley transformation \( \rho_n : \mathbb{H}^n \to \mathbb{B}^n, \rho_n(z, w) = \left( \frac{2z}{1 - iw}, \frac{1 + iw}{1 - iw} \right) \)

  is a biholomorphic mapping between \( \mathbb{H}^n \) and \( \mathbb{B}^n \), we can identify a map \( F \in \text{Prop}_k(\mathbb{B}^n, \mathbb{B}^N) \) or \( \text{Rat}(\mathbb{B}^n, \mathbb{B}^N) \) with \( \rho_N^{-1} \circ F \circ \rho_n \) in the space \( \text{Prop}_k(\mathbb{H}^n, \mathbb{H}^N) \) or \( \text{Rat}(\mathbb{H}^n, \mathbb{H}^N) \) respectively.

  Parametrize \( \partial \mathbb{H}^n \) by \((z, \bar{z}, u)\) through the map \((z, u) \to (z, u + i|z|^2)\). In what follows, we will assign the weight of \( z \) and \( u \) to be 1 and 2, respectively. For a non-negative integer \( m \), a function \( h(z, \bar{z}, u) \) defined over a small ball \( U \) in \( \partial \mathbb{H}^n \) is said to be of quantity \( o_{wt}(m) \) if \( \frac{h(z, \bar{z}, u)}{|\bar{z}|^m} \to 0 \) uniformly for \((z, u)\) on any compact subset of \( U \) as \( t(\in \mathbb{R}) \to 0 \).

- **Partial normalization of \( F \)** Let \( F = (f, \phi, g) = (\bar{f}, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g) \) be a non-constant \( C^2 \)-smooth CR map from \( \partial \mathbb{H}^n \) into \( \partial \mathbb{H}^N \) with \( F(0) = 0 \). For each \( p \in \partial \mathbb{H}^2 \), we write \( \sigma_p^0 \in \text{Aut}(\mathbb{H}^n) \) and \( \tau_p^F \in \text{Aut}(\mathbb{H}^N) \) for the maps

\[
\begin{align*}
\sigma_p^0(z, w) &= (z + z_0, w + w_0 + 2i\langle z, z_0 \rangle), \\
\tau_p^F(z^*, w^*) &= (z^* - \bar{f}(z_0, w_0), w^* - g(z_0, w_0) - 2i\langle z^*, \bar{f}(z_0, w_0) \rangle).
\end{align*}
\]  

(18)

\( F \) is equivalent to \( F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p) \). Notice that \( F_0 = F \) and \( F_p(0) = 0 \). The following is basic for the understanding of the geometric properties of \( F \).

**Lemma 2.1** ([§2, Lemma 5.3, Hu99], [Lemma 2.0, Hu03]): Let \( F \) be a \( C^2 \)-smooth CR map from \( \partial \mathbb{H}^n \) into \( \partial \mathbb{H}^N \), \( 2 \leq n \leq N \) with \( F(0) = 0 \). For each \( p \in \partial \mathbb{H}^n \), there is an automorphism \( \tau_p^{**} \in \text{Aut}_0(\mathbb{H}^N) \) such that \( F_p^{**} := \tau_p^{**} \circ F_p \) satisfies the following normalization:

\[
\begin{align*}
f_p^{**} &= z + \frac{i}{2} a_p^{**}(1)(z) w + o_{wt}(3), \\
\phi_p^{**} &= \phi_p^{**(2)}(z) + o_{wt}(2), \\
g_p^{**} &= w + o_{wt}(4),
\end{align*}
\]

(19)

\[
\langle \bar{z}, a_p^{**}(1)(z) \rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.
\]
Let $A(p) = -2i(\frac{\partial^2(f_p)}{\partial z_j \partial w} |_0)_{1 \leq j, l \leq (n-1)}$. We call the rank of $A(p)$, which we denote by $Rk_F(p)$, the geometric rank of $F$ at $p$. $Rk_F(p)$ depends only on $p$ and $F$, and is a lower semi-continuous function on $p$. We define the geometric rank of $F$ to be $Rk_F := \max_{p \in \partial \mathbb{H}^n} Rk_F(p)$. Notice that we always have $0 \leq Rk_F \leq n - 1$. We define the geometric rank of $F \in \text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ to be the one for the map $\rho_N^{-1} \circ F \circ \rho_\infty \in \text{Prop}_2(\mathbb{H}^n, \mathbb{H}^N)$. It is proved that $F$ is linear fractional if and only if the geometric rank $Rk_F = 0$ ([Theorem 4.3, Hu99]). Hence, in all that follows, we assume that $Rk_F = \kappa_0 \geq 1$.

Denote by $S_0 = \{(j, l) : 1 \leq j \leq \kappa_0, 1 \leq l \leq (n - 1), j \leq l\}$ and write $S := \{(j, l) : (j, l) \in S_0, \text{ or } j = \kappa_0 + 1, l \in \{\kappa_0 + 1, \ldots, \kappa_0 + N - n - \frac{(2n - \kappa_0 - 1)\kappa_0}{2}\}\}$. Then we further have the following normalization for $F$:

**Lemma 2.2 ([Lemma 3.2, Hu03]):** Let $F$ be a $C^2$-smooth CR map from an open piece $M \subset \partial \mathbb{H}^n$ into $\partial \mathbb{H}^N$ with $F(0) = 0$ and $Rk_F(0) = \kappa_0$. Let $P(n, \kappa_0) = \frac{2n(2n - \kappa_0 - 1)}{2}$. Then $N \geq n + P(n, \kappa_0)$ and there are $\sigma \in \text{Aut}_0(\partial \mathbb{H}^n)$ and $\tau \in \text{Aut}_0(\partial \mathbb{H}^N)$ such that $F_{**} = \tau \circ F \circ \sigma := (f, \phi, g)$ satisfies the following normalization conditions:

\[
\begin{align*}
    f_j &= z_j + \frac{\mu_j}{2} z_j w + o_w(3), \quad \frac{\partial^2 f_j}{\partial w^2}(0) = 0, \quad j = 1 \cdots, \kappa_0, \quad \mu_j > 0, \\
    f_j &= z_j + o_w(3), \quad j = \kappa_0 + 1, \cdots, n - 1, \\
    g &= w + o_w(4), \\
    \phi_{jl} &= \mu_{jl} z_j z_l + o_w(2), \quad \text{where } (j, l) \in S \text{ with } \mu_{jl} > 0 \text{ for } (j, l) \in S_0 \\
    \text{and } \mu_{jl} &= 0 \text{ otherwise.}
\end{align*}
\]  

(20)

Moreover $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j, l \leq \kappa_0, j \neq l$, $\mu_{jl} = \sqrt{\mu_j}$ if $j \leq \kappa_0$ and $l > \kappa_0$ or if $j = l \leq \kappa_0$.

Here we denote $\text{Aut}_0(\partial \mathbb{H}^n) = \{\psi \in \text{Aut}(\partial \mathbb{H}^n) \mid \psi(0) = 0\}$.

**Degree of a rational map** For a rational holomorphic map $H = \frac{(P_1, \ldots, P_m)}{Q}$ over $\mathbb{C}^n$, where $P_j, Q$ are holomorphic polynomials and $(P_1, \ldots, P_m, Q) = 1$, we define

$$
\text{deg}(H) = \max\{\text{deg}(P_j), 1 \leq j \leq m, \text{deg}(Q)\}.
$$

For a rational map $H$ and a complex affine subspace $S$ of dimension $k$, we say that $H$ is linear fractional along $S$, if $S$ is not contained in the singular set of $H$ and for any linear parametrization $z_j = z_j^0 + \sum_{l=1}^k a_{jl} t_l$ of $S$ with $j = 1, \cdots, n$, $H^k(t_1, \cdots, t_k) := H(z_1^0 + \sum_{l=1}^k a_{1l} t_l, \cdots, z_n^0 + \sum_{l=1}^k a_{nl} t_l)$ has degree 1 in $(t_1, \cdots, t_k)$.
• Actions of the isotropic groups of the Heisenberg hypersurfaces  Recall from
[(2.4.1), Hu03] and [(2.4.2), Hu03], we define $\sigma \in \text{Aut}_{0}(\partial \mathbb{H}^2)$ and $\tau^* \in \text{Aut}_{0}(\partial \mathbb{H}^5)$ by
\[
\sigma = \left( \frac{\lambda(z + aw) \cdot U}{q(z, w)}, \ \lambda^2 w \right), \quad \tau^*(z^*, w^*) = \left( \frac{\lambda^* (z^* + a^* w^*) \cdot U^*}{q^*(z^*, w^*)}, \lambda^2 w^* \right),
\]
with $q(z, w) = 1 - 2i(\bar{a}, \bar{z}) + (r - i|a|^2)w$, $\lambda > 0$, $r \in \mathbb{R}$, $a, U \in \mathbb{C}$, $|U| = 1$, and $q^*(z^*, w^*) = 1 - 2i(\bar{a}, \bar{z}) + (r^* - i|a|^2)w^*$, $\lambda^* > 0$, $r^* \in \mathbb{R}$, $a^* = (a^*_1, a^*_2) \in \mathbb{C} \times \mathbb{C}$ and $U^*$ is an $4 \times 4$ unitary matrix, such that $[(2.5.1), (2.5.2), Hu03]$ holds:
\[
\lambda^* = \lambda^{-1}, \ a^*_1 = -\lambda^{-1}aU, \ a^*_2 = 0, \ r^* = -\lambda^{-2}r, \ U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U_{22}^{-1} \end{pmatrix},
\]
where $a^* = (a^*_1, a^*_2)$, $U_{22}^*$ is an $3 \times 3$ unitary matrix. Define $F^* = \tau^* \circ F \circ \sigma$. By [Lemma 2.3(A), Hu03], we can write
\[
f(z, w) = z + \frac{i}{2}zAw + o_w(3), \quad f^*(z, w) = z + \frac{i}{2}zA^*w + o_w(3),
\]
\[
\phi(z, w) = \frac{1}{2}z(B^1, B^2, B^3)z + zBw + \frac{1}{2}\partial w(0)w^2 + o((z, w)^2),
\]
\[
\phi^*(z, w) = \frac{1}{2}z(B^1, B^2, B^3)z + zB^*w + \frac{1}{2}\partial w(0)w^2 + o((z, w)^2),
\]
where $B^i = \frac{\partial^2 \phi^*}{\partial z^i} (0)$, $B^i = \frac{\partial^2 \phi^*}{\partial z^i} (0)$ for $i = 1, 2, 3$ and $B = (\frac{\partial^2 \phi^*}{\partial z \partial w}, \frac{\partial^2 \phi^*}{\partial w \partial w}, \frac{\partial^2 \phi^*}{\partial z^2})$, $B^* = (\frac{\partial^2 \phi^*}{\partial z \partial w}, \frac{\partial^2 \phi^*}{\partial w \partial w}, \frac{\partial^2 \phi^*}{\partial z^2})$. Also, the same computation in [Hu03, Lemma 2.3 (A)] gives the following:
\[
\frac{\partial^2 \phi^*}{\partial z^i} (0) = 0, \quad \frac{\partial^2 \phi^*}{\partial z \partial w} (0) = 0, \quad \frac{\partial^2 \phi^*}{\partial w \partial w} (0) = 0, \quad \frac{\partial^2 \phi^*}{\partial z^2} (0) = 0, \quad A^* = \lambda^2 UAU^{-1},
\]
\[
\frac{\partial^2 \phi^*}{\partial z \partial w} (0) = i\lambda^2 aU^2AU^{-1} + \lambda^3 \frac{\partial f}{\partial z^w} (0)U^{-1},
\]
\[
[B^1, B^2, B^3] = \lambda U[B^1, B^2, B^3]U_{22}^*,
\]
\[
[B^1, B^2, B^3] = \lambda U[B^1, B^2, B^3]U_{22}^* + \lambda^2 UB^2U_{22}^*,
\]
\[
\frac{\partial^2 \phi^*}{\partial z \partial w} (0) = \lambda aU[B^1, B^2, B^3]U_{22}^* + 2\lambda^2 aUB^2U_{22}^* + \lambda^2 \frac{\partial^2 \phi^*}{\partial w \partial w} (0)U_{22}^*.
\]

Lemma 2.3 ([HJ06, theorem 4.1]) Let $F \in \text{Rat}(\partial \mathbb{H}^2, \partial \mathbb{H}^N)$ have degree 2 with $F(0) = 0$ and $\text{Rk}F(0) = 1$ ($N \geq 4$). Then
(1) $F$ is equivalent to $(F^{***}, 0)$ where $F^{***} = (f, \phi_1, \phi_2, \phi_3, g) \in \text{Rat}(\partial \mathbb{H}^2, \partial \mathbb{H}^5)$ defined by
\[
f(z, w) = \frac{z - 2bw^2 + (\frac{1}{2} + i \epsilon_1)zw}{1 + iw + ew^2 - 2bw},
\]
\[
\phi_1(z, w) = \frac{1 + iw + ew^2 - 2bw}{ez^2 + ev^2 - 2bw},
\]
\[
\phi_2(z, w) = \frac{1 + iw + ew^2 - 2bw}{ez^2 + ev^2 - 2bw},
\]
\[
\phi_3(z, w) = \frac{1 + iw + ew^2 - 2bw}{ez^2 + ev^2 - 2bw},
\]
\[
g(z, w) = \frac{1 + iw + ew^2 - 2bw}{1 + iw + ew^2 - 2bw}.
\]
Here $b, -e_1, -e_2, c_1, c_2, c_3$ are real non-negative numbers satisfying

$$e_1e_2 = c_2^2 + c_3^2, \quad -e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2, \quad -be_2 = c_1c_2, \quad c_3 = 0 \text{ if } c_1 = 0.$$  
(26)

(2) $c_1, c_2, c_3, e_1, e_2, b$ are uniquely determined by $F$. Conversely, for any non-negative real numbers $c_1, c_2, c_3, e_1, e_2, b$ satisfying the relations in (26), the map $F$ defined in (25) is an element in $\text{Rat}(\partial \mathbb{H}^2, \partial \mathbb{H}^5)$ of degree 2 with $F(0) = 0$ and $\text{Rk}_F(0) = 1$.

Remarks (i) The new normalized map in Lemma 2.3(1) can be obtained by $F^{***} = \tau^* \circ F^{**} \circ \sigma$ where $F^{**}$ is as in Lemma 2.2 and $\sigma$ and $\tau^*$ are as in (21).

(ii) For any map $F$ in Lemma 2.3(1), $b = -e_1 - e_2 - \frac{1}{4} - c_1^2$ and $c_2 = \sqrt{e_1e_2 - c_3^2}$ are determined by $c_1, c_3, e_1$ and $e_2$. Then $c_1, c_3, e_1$ and $e_2$ can be regarded as parameters, and we denote $F = F_{c_1, c_3, e_1, e_2}$.

(iii) We denote by $\mathcal{K}$ a subset of $\mathbb{R}^4$ such that $(c_1, c_3, e_1, e_2)$, or $F_{c_1, c_3, e_1, e_2} \in \mathcal{K}$ if and only if $F_{c_1, c_3, e_1, e_2}$ is a map as above.

Lemma 2.4 ([CJX06, Lemma 2.5]) Let $F \in \text{Rat}(\partial \mathbb{H}^2, \partial \mathbb{H}^5)$ with $F(0) = 0$ and $\deg(F) = 2$. Suppose that $p_m \in \partial \mathbb{H}^2$ is a sequence converging to $0 \in \partial \mathbb{H}^2$ and $F_{p_m}$ is of rank 1 at 0 for any $m$ and $F_{p_m}^{***}$ converges such that $rac{\partial^2 \phi_{1,m}^{***}}{\partial z \partial w} |_0$, $\frac{\partial^2 \phi_{2,m}^{***}}{\partial w^2} |_0$, $\frac{\partial^2 \phi_{3,m}^{***}}{\partial z \partial w} |_0$ and $\frac{\partial^2 \phi_{4,m}^{***}}{\partial w^2} |_0$ are bounded for all $m$. Then

(i) $F$ is of rank 1 at 0.

(ii) $F^{***} \to F^{***}$.

(iii) If we write $F_{p_m}^{***} = G_{2,m} \circ \tau_{p_m} \circ F \circ \sigma_{p_m} \circ G_{1,m}$ where $\sigma_{p_m}$ and $\tau_{p_m} = \tau_{p_m}^F$ are as in (18), $G_{1,m}$ and $G_{2,m}$ are as in (21), then $G_{1,m}$ and $G_{2,m}$ are convergent to some $G_1 \in \text{Aut}_0(\partial \mathbb{H}^2)$ and $G_2 \in \text{Aut}_0(\partial \mathbb{H}^2)$ respectively.

Let $F$ be as in Lemma 2.3 (1). By Lemma 2.3, $F_p$ is equivalent to a map of the following form $F_{p}^{***} = (f_{p}^{***}, \phi_{1,p}^{***}, \phi_{2,p}^{***}, \phi_{3,p}^{***}, \phi_{4,p}^{***}, g_{p}^{***})$ for any $p \in \partial \mathbb{H}^2$ where $\text{Rk}_F(p) = 1$:

$$f_{p}^{***}(z, w) = \frac{z - 2ib(p)z^2 + (\frac{1}{2} + ie_1(p))zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$\phi_{1,p}^{***}(z, w) = \frac{z^2 + b(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$\phi_{2,p}^{***}(z, w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$\phi_{3,p}^{***}(z, w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$\phi_{4,p}^{***}(z, w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$g_{p}^{***}(z, w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z}.$$

$$\phi_{4,p}^{***}(z, w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$g_{p}^{***}(z, w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z}.$$
Proof: (1) All these formulas were proved in [CJX06, lemma 3.1].

Then the followings hold.

Lemma 2.5 Let $F$ and $F_p^{***}$ be as above. Let $p = (z_0, w_0) = (z_0, u_0 + i|z_0|^2) \in \partial \mathbb{H}^2$ near 0.
Then the followings hold.

(i) The real analytic functions have the formulas

$$b^2(p) = b^2 - 4b(2e_1 + c_1)\Im(z_0) + o(1),$$
$$c_1^2(p) = c_1^2 + 4c_1(bc_1 + 2c_2)\Im(z_0) + o(1),$$
$$e_2(p) + e_1(p) = e_2 + e_1 + 8b(e_1 + e_2)\Im(z_0) + o(1),$$
$$c_2^2(p) - e_1(p) - e_2(p) = c_1^2 - e_1 - e_2 + \left(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)\right)\Im(z_0) + o(1)$$

where we denote $o(k) = o(|(z_0, u_0)|^k)$.

(ii) If $c_1 > 0$, the real analytic function has the formula

$$c_3^2(p) = c_3^2 + 4(c_3)^2(5b - \frac{2c_2}{c_1})\Im(z_0) + o(1),$$

(iii) If $c_1 = 0$, then $c_3(p) \equiv 0$.

Proof: (1) All these formulas were proved in [CJX06, lemma 3.1].

(ii) We use the formulas in [CJX06, Step 3 and 4, § 5] and the notation to obtain

$$c_3^2 = \left|\frac{1}{2} \frac{\partial^2 \phi_{33}^{***}}{\partial w^2}(0)\right|^2 = \left|\frac{1}{2} \frac{\partial^2 \phi_{33}^{**}}{\partial w^2}(0)\right|^2 = c_3^2 + 4(c_3)^2(5b - \frac{2c_2}{c_1})\Im(z_0) + o(1).$$

(iii) If $c_1 = 0$, by Lemma 2.3, $c_3 = 0$ and $F \in \text{Rat} (\mathbb{H}^2, \mathbb{H}^4)$. We modify slightly on the normalization $F^{***}$ so that $\phi_{33}^{***} \equiv 0$ and hence $c_3(p) \equiv 0$. □
3 A Monotone Lemma

Recall that for any \((c_1, c_3, e_1, e_2) \in \mathcal{K}\), we denote

- \((c_1, c_3, e_1, e_2) \in \mathcal{K}_I\) (i.e. \(F_{c_1,c_3,e_1,e_2}\) is of the form of type (I)) if \(c_1 = 0\) and \(b = 0\);
- \((c_1, c_3, e_1, e_2) \in \mathcal{K}_II\) (i.e. \(F_{c_1,c_3,e_1,e_2}\) is of the form of type (II)) if \(c_1 > 0\) and \(b = c_2 = 0\).

Also recall that for any map \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{II}\), we denote

- \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{IIA}\) (i.e. \(F_{c_1,c_3,e_1,e_2}\) is of the form of type (IIA)) if \(c_1 > 0\), \(b = c_2 = 0\) and \(c_3 = e_2 = 0\);
- \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{IIB}\) (i.e. \(F_{c_1,c_3,e_1,e_2}\) is of the form of type (IIB)) if \(c_1 > 0\), \(b = c_2 = 0\) and \(c_3 = e_1 = 0\);
- \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{IIC}\) (i.e. \(F_{c_1,c_3,e_1,e_2}\) is of the form of type (IIC)) if \(c_1 > 0\), \(b = c_2 = 0\) and \(c_3 > 0\).

For any \((c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II}\), we denote

- \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,I,1+4e_2+2c_1^2>0}\), if \(1 + 4e_2 + 2c_1^2 > 0\);
- \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,I,1+4e_2+2c_1^2=0}\), if \(1 + 4e_2 + 2c_1^2 = 0\);
- \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,I,1+4e_2+2c_1^2<0}\), if \(1 + 4e_2 + 2c_1^2 < 0\).

For any \(F_{c_1,c_3,e_1,e_2} \in \mathcal{K}\), we define \(\mathcal{W}(F_{c_1,c_3,e_1,e_2}) := \mathcal{W}(c_1, c_3, e_1, e_2) := c_1^2 - e_1 - e_2\). We also consider curves

\[
\Gamma(t) = (\alpha t, \beta t + i|\alpha|^2 t^2) \in \partial \mathbb{H}^2, \quad \forall t \in [0, 1], \quad |\alpha| \leq 1 \text{ and } |\beta| \leq 1
\]

where \(\alpha = \alpha_1 + i\alpha_2, \alpha_j, \beta_1\) are real numbers.

Lemma 3.1 Let \(\Gamma\) be any curve as in (27).

(a) If \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,I,1+4e_2+2c_1^2>0}\), then there exists \(\delta = \delta(\Gamma) > 0\) such that

\[
\mathcal{W}((F_{c_1,c_3,e_1,e_2})_{\Gamma(t)}) \leq \mathcal{W}((F_{c_1,c_3,e_1,e_2})_{\Gamma(t')}), \quad \forall 0 \leq t_1 < t_2 \leq \delta.
\]

(b) If \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,I,1+4e_2+2c_1^2=0}\), then there exists \(\delta = \delta(\Gamma) > 0\) such that

\[
\mathcal{W}((F_{c_1,c_3,e_1,e_2})_{\Gamma(t)}) \equiv \text{constant}, \quad \forall t.
\]

(c) If \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,I,1+4e_2+2c_1^2<0}\), then there exists \(\delta = \delta(\Gamma) > 0\) such that

\[
\mathcal{W}((F_{c_1,c_3,e_1,e_2})_{\Gamma(t)}) \geq \mathcal{W}((F_{c_1,c_3,e_1,e_2})_{\Gamma(t')}), \quad \forall 0 \leq t_1 < t_2 \leq \delta.
\]
Proof of Lemma 3.1: Step a. The basic setup. The monotonicity (28) in (a) means
\[
\frac{d\mathcal{W}(F_{\Gamma(t)}^{***})}{dt} = \lim_{\Delta t \to 0} \frac{\mathcal{W}(F_{\Gamma(t+\Delta t)}^{***}) - \mathcal{W}(F_{\Gamma(t)}^{***})}{\Delta t} \geq 0, \quad \forall t \in [0, \delta]. \tag{31}
\]
For any $0 < t < \delta$ and sufficiently small $\Delta t > 0$, if we can write
\[
F_{\Gamma(t+\Delta t)}^{***} = \left( F_{\Gamma(t)}^{***} \right)_{q(t,\Delta t)}^{***}
\]
for some differentiable map $q(t,\Delta t) \in \partial \mathbb{H}^2$, then from Lemma 2.5 we should have
\[
\mathcal{W}(F_{\Gamma(t+\Delta t)}^{***}) = \mathcal{W}(F_{\Gamma(t)}^{***}) + \left[ 4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2) \right] (\Gamma(t)) \Im(q_1(t)) \Delta t + o(|\Delta t|),
\]
where we write $q(t,\Delta t) := (q_1(t), q_2(t)) \Delta t + o(|\Delta t|)$. Notice that $[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(t)) \geq 0$ always holds because $c_1, c_2, -e_1 - e_2 \geq 0$. Then (31) follows if $\Im(q_1(t)) \geq 0$ holds. In particular, if $[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(t)) \neq 0$ for any fixed $t \in [0, \delta]$, and if
the following condition is satisfied:
\[
\Im(q_1(t)) > 0, \quad \forall t \in [0, \delta],
\tag{34}
\]
then the strict inequality (31) holds. To prove (31), it suffices to prove (34).

Step b. $\Gamma(t)$ determines $q(t,\Delta t)$. To prove (32), we define $q(t,\Delta t)$ by
\[
\Gamma(t + \Delta t) = \sigma_{\Gamma(t)} \circ G_1(q(t,\Delta t)) \tag{35}
\]
where $G_1 = G_1(t) \in \text{Aut}_0(\partial \mathbb{H}^2)$ and $G_2 \in \text{Aut}_0(\partial \mathbb{H}^5)$ are defined such that
\[
(F_{\Gamma(t)})^{***} = G_2 \circ \tau_{\Gamma(t)}^F \circ F \circ \sigma_{\Gamma(t)} \circ G_1.
\tag{36}
\]
By (35), $q(t,\Delta t)$ is a function uniquely determined by $\Gamma(t)$ given by
\[
q(t,\Delta t) = G_1^{-1} \circ \sigma_{\Gamma(t)}^{-1} \circ \Gamma(t + \Delta t).
\tag{37}
\]
The definition (37) will be justified in Step c. Here we derive a formula (39).

By the definition of $\sigma$ (see (18)),
\[
\sigma_{\Gamma(t)}^{-1}(z, w) = (z - z(t), w - w(t) - 2i\langle z, z(t) \rangle + 2i|z(t)|^2),
\]

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and
\[ \Gamma(t + \Delta t) = \left( \alpha(t + \Delta t), \beta_1(t + \Delta t) + i|\alpha|^2(t^2 + 2t\Delta t + \Delta t^2) \right) \]
\[ = \Gamma(t) + (\alpha, \beta_1 + i|\alpha|^2(2t + \Delta t))\Delta t = \Gamma(t) + (\alpha\Delta t, (\beta_1 + 2i|\alpha|^2t)\Delta t) + o(|\Delta t|). \] \tag{38}

Then
\[ \sigma_{\Gamma(t)}^{-1} \circ \Gamma(t + \Delta t) = (\alpha\Delta t, \beta_1\Delta t) + o(|\Delta t|). \]

We denote \( G_1 \in \text{Aut}_0(\partial \mathbb{H}^2) \) as in (21), and we have
\[ G_1(z, w) = \left( \frac{\lambda(z + \bar{a}w)U}{1 - 2i\langle \bar{a}, z \rangle - (r + i|\bar{a}|^2)w}, \frac{\lambda^2w}{1 - 2i\langle \bar{a}, z \rangle - (r + i|\bar{a}|^2)w} \right) \]
where \( U = U(t) = e^{i\theta}, \theta = \theta(t) \in \mathbb{R}, \lambda = \lambda(t) > 0 \) and \( \bar{a} = \bar{a}(t) \in \mathbb{C}, \) and \( r = r(t) \in \mathbb{R}, \) and
\[ G_1^{-1}(z^*, w^*) = \left( \frac{1}{\lambda^2}(z - \bar{a}w)U^{-1}}{1 + 2i\langle \bar{a}, z \rangle + (\frac{1}{\lambda^2}r - i|\bar{a}|^2)w}, \frac{1}{\lambda^2}w \right). \]

Therefore
\[ q(t, \Delta t) = G_1^{-1} \circ \sigma_{\Gamma(t)}^{-1} \circ \Gamma(t + \Delta t) = G_1^{-1}(\alpha\Delta t, \beta_1\Delta t) + o(|\Delta t|) \]
\[ = \left( \frac{1}{\lambda^2}(\lambda tU^{-1} - \bar{a}t\beta_1), \frac{1}{\lambda^2}t\beta_1 \right)\Delta t + o(|\Delta t|). \]

By using the notation in (34), we have
\[ \Im(q_1(t)) = \frac{1}{\lambda(t)^2} \Im \left( \lambda(t)\alpha U(t)^{-1} - \bar{a}(t)\beta_1 \right). \] \tag{39}

**Step c. The identity** We want to prove that the identity (32) holds:
\[ (F_{\Gamma(t+\Delta t)})^{***} = \left( (F_{\Gamma(t)})^{***} \right)_{q(t,\Delta t)}, \] \tag{40}
for sufficiently small \( t \) and \( \Delta t, \) i.e., to prove the following identity
\[ G_4 \circ \tau_{\Gamma(t+\Delta t)}^F \circ F \circ \sigma_{\Gamma(t+\Delta t)} \circ G_3 = G_6 \circ \tau_{q(t,\Delta t)}^F \circ \left( G_2 \circ \tau_{\Gamma(t)}^F \circ F \circ \sigma_{\Gamma(t)} \circ G_1 \right) \circ \sigma_{q(t,\Delta t)} \circ G_5. \] \tag{41}
Here by abusing of notion, we still use $\tau^F_q$ to denote $\tau^H_q$ where $H = (F_{\Gamma(t)})^{***}$. Notice that $G_1, G_5, G_3 \in Aut_0(\partial B_2)$, $\sigma_{\Gamma(t)}, \sigma_q, \sigma_{\Gamma(t)+\Delta t} \in Aut(\partial B_2)$, and $G_2, G_6, G_4 \in Aut_0(\partial B_5)$, $\tau^F_{\Gamma(t)}, \tau^F_q, \tau^F_{\Gamma(t)+\Delta t} \in Aut(\partial B_5)$ are uniquely determined by $F, \Gamma(t), q$ and $\Gamma(t+\Delta t)$ in the normalization process, respectively.

If we can write
\[
\left( \left( (F_{\Gamma(t)})^{***} \right)_{q(t,\Delta t)} \right)^{***} = B \circ (F_{\Gamma(t+\Delta t)})^{***} \circ A
\]
for some $A \in Aut_0(\partial B^2)$ and $B \in Aut_0(\partial B^5)$, then (40) holds by Lemma 2.3(2).

In fact, we write
\[
\left( \left( (F_{\Gamma(t)})^{***} \right)_{q(t,\Delta t)} \right)^{***} =
G_6 \circ \tau^F_q \circ \left( G_2 \circ \tau^F_{\Gamma(t)} \circ F \circ \sigma_{\Gamma(t)} \circ G_1 \right) \circ \sigma_{q(t,\Delta t)} \circ G_5
\]
\[
= \left( G_6 \circ \tau^F_q \circ G_2 \circ \tau^F_{\Gamma(t)} \circ (\tau^F_{\Gamma(t)+\Delta t})^{-1} \circ G_4^{-1} \right) \circ \left( G_4 \circ \tau^F_{\Gamma(t+\Delta t)} \circ F \circ \sigma_{\Gamma(t+\Delta t)} \circ G_3 \right) \circ
\circ \left( G_3^{-1} \circ \sigma_{\Gamma(t+\Delta t)} \circ \sigma_{\Gamma(t)} \circ G_1 \circ \sigma_{q(t,\Delta t)} \circ G_5 \right)
\]
\[
= B \circ (F_{\Gamma(t+\Delta t)})^{***} \circ A
\]
where $B = G_6 \circ \tau^F_q \circ G_2 \circ \tau^F_{\Gamma(t)} \circ (\tau^F_{\Gamma(t)+\Delta t})^{-1} \circ G_4^{-1}$ and $A = G_3^{-1} \circ \sigma_{\Gamma(t+\Delta t)} \circ \sigma_{\Gamma(t)} \circ G_1 \circ \sigma_{q(t,\Delta t)} \circ G_5$.

Writing $A = G_3^{-1} \circ \left( \sigma_{\Gamma(t+\Delta t)} \circ \sigma_{\Gamma(t)} \circ G_1 \circ \sigma_{q(t,\Delta t)} \right) \circ G_5$. Notice $G_3^{-1}, G_5 \in Aut_0(\partial B^2)$. By (35), we know $\sigma_{\Gamma(t+\Delta t)} \circ \sigma_{\Gamma(t)} \circ G_1 \circ \sigma_{q(t,\Delta t)} \in Aut_0(\partial B^2)$. Then $A \in Aut_0(\partial B^2)$. Similarly, we can show $B \in Aut_0(\partial B^5)$.

**Step d. Proof of (a) - the case $\alpha \neq 0$** Let $\alpha$ be as in (39). Suppose $\alpha \neq 0$. By our construction (see [CJX06, Step 3 in § 5]), the vector $\bar{a}$ and the matrix $U$ in (39) are given by

\[
\bar{a} = \bar{a}(t) = i \frac{\partial^2 F^{**}}{\partial w^2}(0) = i(e_1 - 2e_2)z_0 + 2ic_1c_2u_0 + (|p|) = i(e_1 - 2e_2)\alpha t + o(t),
\]
\[
U = U(t) = \begin{cases} 
  e^{i\theta} \frac{\partial^2 \phi^{**}_{\mu \nu}}{\partial z \partial w}(0)/\left| \frac{\partial^2 \phi^{**}_{\mu \nu}}{\partial z \partial w}(0) \right|, & \text{if} \quad \frac{\partial^2 \phi^{**}_{\mu \nu}}{\partial z \partial w}(0) \neq 0, \\
  1, & \text{if} \quad \frac{\partial^2 \phi^{**}_{\mu \nu}}{\partial z \partial w}(0) = 0,
\end{cases}
\]
and (see [CJX06, Step 3 in § 5])

\[
\frac{\partial^2 \phi_{p1}^{**}}{\partial z \partial w}(0) = \frac{\partial^2 \phi_{p1}^{**}}{\partial z \partial w}(0) = b - 2ib^3u_0 - ibc_1u_0 - 4ib^2z_0 - i\frac{b}{2}bu_0 \\
-iz_0 - 4ie_2z_0 + 4ic_1c_2u_0 - 2ibc_1^2u_0 - 2ic_1^2z_0 = -i(1 + 4e_2 + 2c_1^2)z_0 + o(|p|),
\]

where \(p = (z_0, w_0) = \Gamma(t) = (\alpha t, \beta t + i|\alpha|^2t^2) \in \partial \Omega^2\). Here we used the fact that \(b = c_2c_1 = 0\) because \((c_1, c_3, e_1, e_2) \in K_I \cup K_{II}\). Then we obtain

\[
\frac{\partial^2 \phi_{p1}^{**}}{\partial z \partial w}(0) = -i(1 + 4e_2 + 2c_1^2)\alpha t + o(t)
\]

(45)

Now \(1 + 4e_2 + 2c_1^2 > 0\). Since \(\alpha \neq 0\), we have \(\frac{\partial^2 \phi_{p1}^{**}}{\partial z \partial w}(0) \neq 0\) by (45) so that \(\tilde{a}, U^{-1}\) and \(q_1\) are real analytic near 0 from their construction (cf. [CJX06]). Then

\[
U(t)^{-1} = e^{-i\theta} \frac{\frac{\partial^2 \phi_{p1}^{**}}{\partial z \partial w}(0)}{\frac{\partial^2 \phi_{p1}^{**}}{\partial z \partial w}(0)} = i(1 + 4e_2 + 2c_1^2)\alpha t + o(|t|) \frac{\frac{\partial^2 \phi_{p1}^{**}}{\partial z \partial w}(0)}{\frac{\partial^2 \phi_{p1}^{**}}{\partial z \partial w}(0)} = i(1 + 4e_2 + 2c_1^2)\alpha t + o(|t|).
\]

and there exists a constant \(\delta > 0\) such that

\[
\Im(q_1(t)) = \frac{1}{\lambda(t)^2} \Im\left(\lambda(t)\alpha U(t)^{-1} - \tilde{a}(t)\beta_1\right) = \frac{1}{\lambda(t)^2} \Im\left(\alpha U(t)^{-1}\right) + O(t)
\]

\[
= \frac{1}{\lambda} \Im\left(\frac{i(1 + 4e_2 + 2c_1^2)|\alpha|^2}{|1 + 4e_2 + 2c_1^2|\alpha|}\right) + O(|t|) = |\alpha| + O(|t|), \quad \forall t \in [0, \delta]
\]

(46)

because \(\lambda = \lambda(t) = 1 + O(|t|)\). This proves (34) as well as (28).

**Step e. Proof of (a) - the case \(\alpha = 0\)** Next we will prove (a) for the case \(\alpha = 0\). In this case \(\Gamma(t) = (0, \beta t)\), and \(\Im(q_1(t)) = -\frac{\beta}{\lambda(t)^2} \Im(\tilde{a}(t))\) and \(\tilde{a}(t) = i\frac{\partial f_{p1}^{**}}{\partial w}\)(0). From [CJX06, § 5, step 3 and step 2], we have \(\frac{\partial f_{p1}^{**}}{\partial w}(0) = \frac{\partial f_{p1}^{**}}{\partial w^2}(0) =

\[
= \frac{1}{\lambda(p)} T^2 \tilde{f}(p) \cdot \overrightarrow{Lf(p)} - \frac{1}{\lambda(p)^2} (T^2 \tilde{f} \cdot \overrightarrow{Lf}) (T^2 g - 2iT^2 \tilde{f} \cdot \overrightarrow{f} - 2i\|T^2 \tilde{f}\|^2)(p)
\]

(47)

We want to prove \(\tilde{a}(t) \equiv 0\) which implies (28). This will be done by direct computation. Write \(F\) as in the following form:

\[
f = zh + \left(\frac{i}{2} + ie_1\right)zwh, \phi_1 = z^2h, \phi_2 = c_1zwh, \phi_3 = c_3w^2h, g = wh + ie_1w^2h,
\]
where \( h = h(w) = \frac{1}{1 + i e_1 w + e_2 w^2} \). Then
\[
h' = (-ie_1 - 2e_2 w)h^2, \quad h'' = (-2e_2 - 2e_1^2 + 6ie_1 e_2 w + 6e_2^2 w^2)h^3.
\]

From the definition of \( F_p \) where \( p = (z, w) \), we have [CJH06, § 5]

\[
f(p) = zh + \left( \frac{i}{2} + ie_1 \right) zwh,
\]

\[
Lf(p) = h + \left( \frac{i}{2} + ie_1 \right) zh + 2i\overline{z} \left( zh' + \left( \frac{i}{2} + ie_1 \right) z(h + wh') \right),
\]

\[
Tf(p) = zh' + \left( \frac{i}{2} + ie_1 \right) z(h + wh'),
\]

\[
T^2 f(p) = zh'' + \left( \frac{i}{2} + ie_1 \right) z(2h' + wh''),
\]

\[
\phi_1(p) = z^2 h, \quad L\phi_1(p) = 2zh + 2i\overline{z}z^2 h', \quad T\phi_1(p) = z^2 h',
\]

\[
\phi_2(p) = c_1 zwh, \quad L\phi_2(p) = c_1 wh + 2ic_1 \overline{z} z(h + wh'), \quad T\phi_2(p) = c_1 z(h + wh'),
\]

\[
T^2 \phi_1(p) = z^2 h'',
\]

\[
L^2 \phi_2(p) = 2ic_1 \overline{z}(h + wh') + 2i\overline{z} \left[ c_1 (h + wh') + 2ic_1 \overline{z} (2h' + wh'') \right]
= 4ic_1 \overline{z}(h + wh') - 4c_1 \overline{z}z(2h' + wh''),
\]

\[
T^2 \phi_2(p) = c_1 z(2h' + wh''),
\]

\[
\phi_3(p) = c_3 w^2 h, \quad L\phi_3(p) = 2ic_3 \overline{z}(2wh + w^2 h'), \quad T\phi_3(p) = c_3 (2wh + w^2 h'),
\]
\[ T^2\phi_3(p) = c_3(2h + 2wh' + 2wh'' + w^2h'') = c_3(2h + 4wh' + w^2h''), \]

When \( p = (0,t) \), we have

\[ \lambda(p) = |Lf(p)|^2 + |L\phi_1(p)|^2 + |L\phi_2(p)|^2 + |L\phi_3(p)|^2 = |h(t)|^2 + |c_1th(t)|^2 = 1 + o(t) \]

and \( Tf(p) = T\phi_1(p) = T\phi_2(p) = L\phi_3(p) = T^2f(p) = T^2\phi_1(p) = T^2\phi_2(p) = 0 \) so that

\( (T \tilde{f} \cdot L\tilde{f})(p) = 0 \) and that \( (T^2 f \cdot Lf)(p) = 0 \). Hence by (47) we obtain \( \Im(q_1(t)) = \beta, \frac{\partial}{\partial t}\Im(\tilde{a}(t)) = 0 \). The proof of (a) is complete.

**Step f. Proof of (b) and (c)** Similarly we can prove (c). To prove (b), we first consider the case when \( \alpha \neq 0 \). In this case, we can take a sequence of points \((c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \in \mathcal{K}_{IIC,1+4e_2+2c_1^2,0} \) such that \((c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \rightarrow (c_1, c_3, e_1, e_2) \). Then (46) holds for such maps \( F_{c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}} = \mathcal{K}_{IIC,1+4e_2+2c_1^2,0} \):

\[ \Im(q_1^{(k)}(t))) = |\alpha| + O(|t|), \quad \forall t \in [0, \delta] \quad (48) \]

Also, we can take another sequence of points \((c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \in \mathcal{K}_{IIC,1+4e_2+2c_1^2,0} \) such that \((c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \rightarrow (c_1, c_3, e_1, e_2) \). Then by letting \( k \rightarrow \infty \) and the same argument in the proof for (c), we get

\[ \Im(q_1^{(k)}(t))) = -|\alpha| + O(|t|), \quad \forall t \in [0, \delta] \quad (49) \]

for maps \( F_{c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}} \). Such estimate is uniform for all \( k \). Notice that the function

\[ [4c_1(bc_1+2c_2)-8b(e_1+e_2)](\Gamma(t))\Im(q_1(t)) \] in (33) is real analytic but \( 4c_1(bc_1+2c_2)-8b(e_1+e_2) \) and \( \Im(q_1) \) may be not (see Remark (a) following the proof of Lemma 3.1 below). Then by (48) and (49) and by letting \( k \rightarrow \infty \), we must have

\[ [4c_1(bc_1+2c_2)-8b(e_1+e_2)](\Gamma(t))\Im(q_1(t)) = 0, \quad \forall t \in [0, \delta] \]

for the map \( F_{c_1,c_3,e_1,e_2} \) so that \( \Im(q_1(t)) \) is proved.

Next we consider the case when \( \alpha = 0 \), by Step e, we have \( \Im(q_1(t)) \equiv 0 \) so that (c) is proved. \( \square \)

**Remark (a)** We notice that if \( 1 + 4e_2 + 2c_1^2 = 0 \), \( \frac{\partial^2 \phi_{m=1}}{\partial z \partial w}(0) \) may be zero so that \( U^{-1} \) may not be differentiable. By the way, \( \mathcal{W}(F_{p}^{***} = c_1^2(p) - e_1(p) - e_2(p) = \frac{1}{4} + 2c_1^2(p) + b^2(p) \) is real analytic but \( c_1(p) \) and \( b(p) \) may not be differentiable; this is because of some definitions such as (44) (cf. [CJX06, p.1521-1522]). Then the function
and hence (46) holds.

(b) If we replace the curve \( \Gamma(t) = (\alpha t, \beta t + \frac{1}{2} |\alpha|^2 t^2) \) by another curve
\[
\Gamma(t) = (\alpha t, \beta_0 + \beta_1 t + i |\alpha|^2 t^2),
\]
then (38) and hence (46) holds.

Recall \((c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I} \iff (5) \) holds with \( c_1 > 0 \) and \( b = c_2 = 0 \iff c_1 > 0 \) and either
\[
e_1 = \frac{-(\frac{1}{4} + c_1^2) - \sqrt{(\frac{1}{4} + c_1^2)^2 - 4c_3^2}}{2}, \quad e_2 = \frac{-(\frac{1}{4} + c_1^2) + \sqrt{(\frac{1}{4} + c_1^2)^2 - 4c_3^2}}{2},
\]
where \( 4c_3^2 \leq (\frac{1}{4} + c_1^2)^2 \), or
\[
e_1 = \frac{-(\frac{1}{4} + c_1^2) + \sqrt{(\frac{1}{4} + c_1^2)^2 - 4c_3^2}}{2}, \quad e_2 = \frac{-(\frac{1}{4} + c_1^2) - \sqrt{(\frac{1}{4} + c_1^2)^2 - 4c_3^2}}{2},
\]
where \( 4c_3^2 \leq (\frac{1}{4} + c_1^2)^2 \). Here \( c_1 \) and \( c_3 \) are parameters.

We can write a disjoint union \( \mathcal{K}_{I, I} = \mathcal{K}_{I, I, e_1 < e_2} \cup \mathcal{K}_{I, I, e_1 = e_2} \cup \mathcal{K}_{I, I, e_1 > e_2} \), where
\[
\mathcal{K}_{I, I, e_1 < e_2} = \{(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I} \mid e_1 < e_2 \}
\]
\[
\mathcal{K}_{I, I, e_1 = e_2} = \{(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I} \mid e_1 = e_2 \},
\]
and
\[
\mathcal{K}_{I, I, e_1 > e_2} = \{(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I} \mid e_1 > e_2 \}.
\]

Then \( \mathcal{K}_{I, I, e_1 < e_2} = \{(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I} \mid (51) \) and \( 4c_3^2 < (\frac{1}{4} + c_1^2)^2 \) hold\}, \( \mathcal{K}_{I, I, e_1 = e_2} = \{(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I} \mid (51) \) or \( (52) \) and \( 4c_3^2 = (\frac{1}{4} + c_1^2)^2 \) hold\}, and \( \mathcal{K}_{I, I, e_1 > e_2} = \{(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I} \mid (52) \) and \( 4c_3^2 < (\frac{1}{4} + c_1^2)^2 \) hold\}.

**Lemma 3.2** (i) \( \mathcal{K}_{I, I, e_1 < e_2} \subseteq \mathcal{K}_{I, I, I, 1 + 4e_2 + 2c_3^2 > 0} \), and \( \mathcal{K}_{I, I, e_1 = e_2} \subseteq \mathcal{K}_{I, I, 1 + 4e_2 + 2c_3^2 > 0} \).

(ii) Let \( (c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I, e_1 > e_2} \). Then
(a) \( (c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I, I, 1 + 4e_2 + 2c_3^2 > 0} \) if and only if \( \frac{1}{2} c_1^2 + c_3^2 < 4c_3^2 < (\frac{1}{4} + c_1^2)^2 \) holds.
(b) \( (c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I, I, 1 + 4e_2 + 2c_3^2 = 0} \) if and only if \( \frac{1}{2} c_1^2 + c_3^2 = 4c_3^2 \) holds.
(c) \( (c_1, c_3, e_1, e_2) \in \mathcal{K}_{I, I, I, 1 + 4e_2 + 2c_3^2 < 0} \) if and only if \( 0 \leq 4c_3^2 < \frac{1}{2} c_1^2 + c_3^2 \) holds.
Proof of Lemma 3.2: (i) For any \((c_1, c_3, e_1, e_2) \in K_{II, e_1 < e_2} \cup K_{II, e_1 = e_2}\), by \(-e_1 - e_2 = \frac{1}{2} + c_1^2\) and (51), we have

\[
1 + 4e_2 + 2c_1^2 = \frac{1}{2} + 2e_2 - 2e_1 = \frac{1}{2} + 2\sqrt{\left(\frac{1}{4} + c_1^2\right)^2 - 4c_3^2} \geq \frac{1}{2} > 0.
\]

(ii) For any \((c_1, c_3, e_1, e_2) \in K_{II, e_1 > e_2}\), we know that \(1 + 4e_2 + 2c_1^2 > 0\) is equivalent to \(\frac{1}{2} + 2e_2 - 2e_1 = \frac{1}{2} - 2\sqrt{\left(\frac{1}{4} + c_1^2\right)^2 - 4c_3^2} > 0\), i.e., \(\frac{1}{2}c_1^2 + c_3^2 < 4c_3^2\), so that (a) is proved. (b) and (c) are proved similarly. \(\square\).

Lemma 3.3 Let \(\mathcal{E} := \{(c_1, c_3, e_1, e_2) \in K_{II} \cup K_{II} \mid (F_{c_1, c_3, e_1, e_2})^{**} = F_{c_1, c_3, e_1, e_2}, \forall p \in \partial \mathbb{H}^2 \text{ near } 0\}\). Then \(F_{c_1, c_3, e_1, e_2} \in \mathcal{E}\) if and only if for any curve \(\Gamma\) as in (27),

\[
(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2))(\Gamma(t)) \equiv 0, \forall t \in [0, 1].
\]

Proof: It is clear

\[
F_{c_1, c_3, e_1, e_2} \in \mathcal{E} \iff c_1(p), c_3(p) \text{ are constant, } \forall p \in \partial \mathbb{H}^2 \text{ near } 0.
\]

If \(F_{c_1, c_3, e_1, e_2} \in \mathcal{E}\), then either \(c_1(p) = b(p) = 0\) or \(c_1(p) > 0, b(p) = c_2(p) = 0, \forall p \in \partial \mathbb{H}^2\) near 0 (i.e., the case (I) or (IIA), (IIB) and (IIC)). Then the equality in (53) holds.

Conversely, suppose that \((4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2))(\Gamma(t)) \equiv 0\) for any choice of curve \(\Gamma(t)\) and for any \((c_1, c_3)\) in some open subset of \(\mathbb{R}^2\). Then \(b_1(p) = 0\) and \(c_1(p)c_2(p) = 0\), \(\forall p \in \partial \mathbb{H}^2\) near 0. If \(c_1 \equiv 0\), then by Lemma 2.5(iii), \(c_3(p) = 0, \forall p\) so that \(F_{c_1, c_3, e_1, e_2} \in \mathcal{E}\). If \(c_1(p) > 0\) for any \(p\) in some open subset of \(\partial \mathbb{H}^2\), then \(c_2(p) = 0, \forall p\). Then we apply Lemma 2.5(ii) to know

\[
c_3^2(p) = c_3^2 + 4(c_3)^2(5b - \frac{2c_2}{c_1})\Theta(z_0) + o(|p|) = c_3^2 + o(|p|), \text{ where } p = (z_0, w_0) \in \partial \mathbb{H}^2
\]

which implies as in (33) that \(c_3(p) = \text{constant}, \forall p\). Also, by (33), from \((4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2))(\Gamma(t)) \equiv 0\) it implies \(\mathcal{W}((F_{c_1, c_3, e_1, e_2})^{**}) = \text{constant}, \forall \Gamma\) and \(\forall t\). Then

\[
\mathcal{W}((F_{c_1, c_3, e_1, e_2})^{**}) = (c_1^2 - e_1 - e_2)(\Gamma(t)) = \left(\frac{1}{4} + 2c_1^2\right)(\Gamma(t)) = \text{constant},
\]

which implies that \(c_1(\Gamma(t)) = \text{constant for any } t \in [0, t_0]\), i.e., \(c_1 \equiv \text{constant}. \) By (54), we obtain \(F_{c_1, c_3, e_1, e_2} \in \mathcal{E}\). Claim (53) is proved. \(\square\)

Theorem 1.1(i) will follow by Lemma 3.2 and the following lemma.
Lemma 3.4 Let \((c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II}\). Then \(F_{c_1, c_3, e_1, e_2}\) satisfies (9) if and only if \(F_{c_1, c_3, e_1, e_2} \in \mathcal{K}^* := \mathcal{K}_I \cup \mathcal{K}_{II} - \mathcal{K}_{I, II, 1+4e_2+2c_3^2 < 0}\).

\textbf{Proof:} (\(\iff\)) It follows from Lemma 3.1.

\((\implies\)) Take any map \(F_{c_1, c_3, e_1, e_2} \in \mathcal{K}_{I, II, 1+4e_2+2c_3^2 < 0}\) satisfying the minimum property (9). We first show that \(F_{c_1, c_3, e_1, e_2} \in \mathcal{E}\) where \(\mathcal{E}\) was defined in above lemma.

By Step d in the proof of Lemma 3.1, we know that for any curve \(\Gamma\) as in Lemma 3.1, there is \(\delta > 0\) such that
\[
\Im(q_1(t)) = -|\alpha| + O(|t|), \quad \forall t \in [0, \delta].
\]

Suppose that \(F_{c_1, c_3, e_1, e_2}\) satisfies (9). By (33), it implies \((4c_1(bc_1+2c_2)−8b(e_1+e_2))(\Gamma(t)) \equiv 0\) for any such curves \(\Gamma(t)\) and for any \((c_1, c_3)\) with \(0 \leq 4c_3^2 \leq (\frac{1}{4}+c_3^2)^2\). Then by above lemma, \(F_{c_1, c_3, e_1, e_2} \in \mathcal{E}\).

\(\mathcal{E} \cap \mathcal{K}_{I, II, 1+4e_2+2c_3^2 < 0}\) is a real analytic set in \(\mathcal{K}_{I, II, 1+4e_2+2c_3^2 < 0}\). We claim:
\[
\mathcal{E} \cap \mathcal{K}_{I, II, 1+4e_2+2c_3^2 < 0} = \emptyset. \tag{56}
\]

Suppose (56) is not true. Then we can take
\[
(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}_{I, II, 1+4e_2+2c_3^2 < 0} \cap \mathcal{E}. \tag{57}
\]

We can take a sequence of points \((c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \in \mathcal{K}_{I, II, 1+4e_2+2c_3^2 < 0} - \mathcal{E}\) such that
\[
(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \to (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}).
\]

By our choice of \((c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)})\), the corresponding maps \(F_{c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}}\) has the property that the associated function \(W((F_{c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)})_{\Gamma(t)})\) is strictly decreasing as \(t\) goes from 0 to 1. Then \(F_{c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}}\) is equivalent to some map \(F_{c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}} \in \mathcal{K}^* = \mathcal{K}_I \cup \mathcal{K}_{II} - \mathcal{K}_{I, II, 1+4e_2+2c_3^2 < 0}\) with the minimum \(W\) value. Since the function value \(W((F_{c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)})_{\Gamma(t)})\) is decreasing, the sequence of points \((c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)})\) is also bounded in \(\mathcal{K}\). By taking subsequence, we may assume that \((c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \to (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^*\). Then \(F_{c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}}\) is equivalent to \(F_{c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}} \in \mathcal{K}^*\), i.e.,
\[
F_{c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}} \equiv \left(F_{c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}}(q)\right)^{\text{***}}
\]

for some non zero \(q \in \partial \mathbb{H}^2\), by the same argument as in (7) and (8) (or [CJX06, Step 1, § 4]). On the other hand, since \(F_{c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}} \in \mathcal{E}\), by the definition of \(\mathcal{E}\), (58) cannot occur. This contradiction shows that (57) cannot occur. Thus Claim (56) is proved. \(\square\)
4 Local version of Theorem 1.1(ii)

For each point \( p = (a, b + |a|^2) \in \partial \mathbb{H}^2 \) where \( b \in \mathbb{R} \) and \( a \in \mathbb{C} \), we denote \( \pi(p) = \pi(a, b + i|a|^2) = ((|a|, |b|) \in \mathbb{R}^2 \). We denote by \( \square_c := [0, c] \times [0, c] \) a square and \( \triangle_c := \{(x, y) \mid 0 \leq x \leq c, 0 \leq y \leq x \} \). Let \( \Gamma(t) = (\alpha t, \beta_1 t + i|a|^2 t^2) \) with \( t \in [0, 1] \) be line segments. The set \( \{ \pi(\Gamma(t)) = \pi(\alpha t, \beta_1 t + i|a|^2 t^2) \mid |\alpha| = 1, |\beta_1| \leq 1, 0 \leq t \leq t_0 \} \) is equal to \( \Delta_{t_0} \). Notice that \( \pi(a, b + i|a|^2) \in \triangle_{t_0} \) if and only if there exists such a line segment \( \Gamma(t) \) so that \( (a, b + i|a|^2) = \Gamma(t^*) \) for some \( t^* \in [0, t_0] \).

Lemma 4.1 For any \( P^{(0)} = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^* \), there is a neighborhood \( U \) of \( P^{(0)} \) in \( \mathcal{K}^* \) and a constant \( c > 0 \) such that for any point \( (c_1', c_3', e_1', e_2') \in U \) with \( F_{c_1', c_3', e_1', e_2'} = (F_{c_1', c_3', e_1', e_2'})^{***} \) where \( p = (a, b + i|a|^2) \in \partial \mathbb{H}^2 \), \( a \in \mathbb{C}, b \in \mathbb{R}, |p| := \max\{|a|, |b|\} \leq c \), we have

\[
(c_1', c_3', e_1', e_2') = (c_1, c_3', e_1', e_2').
\] (59)

Proof of Lemma 4.1: Step 1. Choose \( U \) and \( c \) For the point \( P^{(0)} \in \mathcal{K}^* \), by Lemma 3.1 and the uniform estimate (46), there exists a neighborhood \( U \) of this point and a constant \( 0 < t_0 < 1 \) such that for any point \( (c_1', c_3', e_1', e_2') \in U \) and for any curve \( \Gamma(t) = \{(\alpha t, \beta_1 t + i|a|^2 t^2) \} \) with \( (c_1, c_3, e_1' e_2') \in \mathcal{K}^* \), \( \beta_1 \in \mathbb{R} \) with \( |\beta_1| \leq 1, |\alpha| = 1, 0 \leq t \leq t_0 \), we have the property

\[
W((F_{c_1', c_3', e_1', e_2'})^{***}) \text{ is nondecreasing, } \forall t \in [0, t_0].
\] (60)

Since \( F_{c_1', c_3', e_1', e_2'}^{***} = (F_{c_1', c_3', e_1', e_2'})^{***} = H \circ \tau \circ F_{c_1', c_3', e_1', e_2'} \circ \sigma_p \circ G \) where \( G \in Aut_0(\partial \mathbb{H}^2), H \in Aut_0(\partial \mathbb{H}^2), \tau \) and \( \sigma_p \) are as in (18), we can write

\[
F_{c_1', c_3', e_1', e_2'}^{***} = (F_{c_1', c_3', e_1', e_2'})^{***}_q,
\]

where \( q = G^{-1}(-z_0, -w_0) \). Since \( G(0) = 0 \) and \( G^{-1}(0) = 0 \), by continuity, \( q \to 0 \) as \( p \to 0 \).

Then we can choose a number \( 0 < c < t_0 \) such that \( \forall p = (a, b + i|a|^2) \in \partial \mathbb{H}^2 \) with \( |p| \leq c \), the point \( q = (A, B + i|A|^2) \) satisfies \( |q| \leq t_0 \). Let us verify that \( c \) is the desired number.

Step 2. There exists a curve from 0 to \( p \) with monotone property We have to put the condition \( |\alpha| = 1 \) in (60); otherwise we may not be able to find the \( t_0 \) for all curves. We want to remove this condition by adding one more piece of the line segment, namely, we claim that for any \( p \) and \( (c_1', c_3', e_1', e_2') \) as above, there is a curve \( \Gamma(t), t \in [0, t^*] \), consisting of one or two pieces of line segments, such that (60) is still true: \( W((F_{c_1', c_3', e_1', e_2'})^{***}) \) is nondecreasing along \( \Gamma \).

Write \( p = (a, b + i|a|^2) \in \partial \mathbb{H}^2 \). We distinguish two cases: (i) \( \pi(a, b + i|a|^2) \in \triangle_c \); and (ii) \( \pi(a, b + i|a|^2) \in \square_c - \triangle_c \).
In the first case (i): for any \( p = (a, b + i|a|^2) \) with \( |a| \leq c \) and \( |b| \leq |a|c \), assuming \( p \neq 0 \), we have \( p = \Gamma(t^*) \) for some curve \( \Gamma(t) = (at, \beta_1 t + i|a|^2t^2) \) with \( 0 \leq \beta_1 \leq 1 \) and \( |a| = 1 \) as above with some \( t^* \in [0, c] \). In fact, we have \( \alpha = \frac{a}{|a|}, \beta_1 = \frac{b}{|a|} \) and \( t^* = |a| \). By (60) the function \( W(F_{c_1',c_3',e_3',e_2'}^{***} \Gamma(t)) \) is increasing as \( t \) varies from 0 to \( t^* \).

In the second case (ii): \( p = (a, b + i|a|^2) \) with \( |a| \leq c \) and \( |a| < |b| \leq c \). Let us assume \( b > 0 \); otherwise it can be proved by the same argument. In this case, we cannot find \( \Gamma \) such that it connects 0 and \( p \) as in the case (i). However, we can define two pieces of curves:

\[
\Gamma(t) = \begin{cases} 
\Gamma_1(t), & 0 \leq t \leq b - |a|, \\
\Gamma_2(t), & b - |a| \leq t \leq b.
\end{cases}
\]

Here \( \pi(\Gamma_1) = \{0\} \times [0, b - |a|] \) is a vertical line segment; and \( \pi(\Gamma_2) \) is another line segment connecting \( \Gamma_1(b - |a|) \) and the point \( p \).

By Step e in § 3, the function \( W(F_{c_1',c_3',e_3',e_2'}^{***} \Gamma_1(t)) \) is constant for \( 0 \leq t \leq b - |a| \). Next we consider \( W(F_{c_1',c_3',e_3',e_2'}^{***} \Gamma_2(t)) \). If we use a new variable \( u = t - b + |a| \), then \( \Gamma_2(t) \) can be written as

\[
\Gamma_2(u) = \left( \frac{a}{|a|}u, (b - |a|) + u + iu^2 \right), \quad 0 \leq u \leq |a|.
\]

By the remark (b) in (50), (46) is still valid for \( \Gamma_2(u) \) so that \( W(F_{c_1',c_3',e_3',e_2'}^{***} \Gamma_2(t)) \) is nondecreasing for any \( b - |a| \leq t \leq t^* \). Our claim is proved.

**Step 3. The \( W \) function is constant**

We claim:

\[
W(F_{c_1',c_3',e_3',e_2'}^{***} \Gamma(t)) = \text{constant}. \quad (61)
\]

In fact, since \( F_{c_1',c_3',e_3',e_2'}^{***} = (F_{c_1',c_3',e_3',e_2'}^{***}p)^q \) and \( F_{c_1',c_3',e_3',e_2'}^{***} = (F_{c_1',c_3',e_3',e_2'}^{***}q)^p \). We have \( F_{c_1',c_3',e_3',e_2'}^{***} = (F_{c_1',c_3',e_3',e_2'}^{***})^{pq} \).

Since \( \pi(p) \in \square_c \), by our choice of \( c, q = (A, B + i|A|^2) \) satisfies \( \pi(q) \in \square_t \), i.e., \( |A| \leq t_0 \) and \( |B| \leq t_0 \). Then by Step 2, there exists a curve \( \tilde{\Gamma}(\tilde{t}) \), \( 0 \leq \tilde{t} \leq \tilde{t}^* \), connecting 0 and \( q \) such that the function \( W(F_{c_1',c_3',e_3',e_2'}^{***} \tilde{\Gamma}(\tilde{t})) \) is nondecreasing along \( \tilde{\Gamma} \). Then we obtain

\[
W(F_{c_1',c_3',e_3',e_2'}) = W(F_{c_1',c_3',e_3',e_2'}^{***}) \leq W(F_{c_1',c_3',e_3',e_2'}^{***} \Gamma(t^*)) = W(F_{c_1',c_3',e_3',e_2'}^{***}), \quad (62)
\]

and

\[
W(F_{c_1'',c_3'',e_3'',e_2''}) = W(F_{c_1'',c_3'',e_3'',e_2''}^{***}) \leq W(F_{c_1'',c_3'',e_3'',e_2''}^{***} \Gamma(t^*)) = W(F_{c_1'',c_3'',e_3'',e_2''}). \quad (63)
\]

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We first claim that for any 

\[ (F_{t}, e_{1}', e_{2}')_{\Gamma(t)}^{***} \equiv F_{t}, e_{1}', e_{2}', \quad \forall t \in [0, t_{0}). \]  

(64)

Let us consider the case (i) in Step 2. From (31) and Lemma 2.5, it implies that 

\[ 4c_{1}'(b'c_{1}' + 2c_{2}') - 8b'(c_{1}' + c_{2}') \Gamma(t) = 0 \text{ for any } t \in [0, t^*]. \]  

Thus by the argument in (55), we proved 

\[ c_{1}'(\Gamma(t)) = c_{3}'(\Gamma(t)) = 0 \text{ for any } t \in [0, t^*]. \]  

This implies that \( (F_{t}, e_{1}', e_{2}')_{\Gamma(t)}^{***} \) is the same map for any \( t \in [0, t_{0}]. \) Claim (64) is proved. The case (ii) will be proved by similar argument as the case (i) and by the remark (b) in (50). \( \square \)

**Lemma 4.2** For any point \( P^{(0)} = (c_{1}^{(0)}, c_{3}^{(0)}, e_{1}^{(0)}, e_{2}^{(0)}) \in \mathcal{K}^{*} - \mathcal{E} \) where \( \mathcal{E} \) is defined in Lemma 3.3, there is a neighborhood \( V \) of \( P^{(0)} \) in \( \mathcal{K}^{*} \), a neighborhood \( U \) of \( P^{(0)} \) in \( \mathcal{K}^{*} - \mathcal{E} \) and a neighborhood \( E \) of 0 in \( \partial \mathbb{H}^2 \) such that the map \( \Psi : U \times E \rightarrow V, (F, p) \mapsto F_{p}^{***} \) is surjective.

**Proof:** We first claim that for any \( F_{c_{1}, c_{3}, e_{1}, e_{2}} \in \mathcal{K}^{*} - \mathcal{E} \), the set \( N := \{(F_{c_{1}, c_{3}, e_{1}, e_{2}})_{p}^{***} \mid p \in \partial \mathbb{H}^2 \} \) is of real dimension \( \geq 2 \). In fact, consider a function \( W(F_{c_{1}, c_{3}, e_{1}, e_{2}})_{\Gamma(t)}^{***} \) on \( N \) where \( \Gamma(t) = (\alpha t, \beta t + |\alpha|^2 t^2) \) is a curve in \( \partial \mathbb{H}^2 \) as (27). By (46), we have \( 3(q_{1}(t)) = |\alpha| + O(|t|) \) for \( t > 0 \) sufficiently small. Since \( F_{c_{1}, c_{3}, e_{1}, e_{2}} \in \mathcal{K}^{*} - \mathcal{E} \), by Lemma 3.3, we have \( (4c_{1}(b c_{1} + 2c_{2}) - 8b(e_{1} + e_{2})) (\Gamma(t)) \neq 0 \) holds for some curve \( \Gamma \). Then from (33),

\[ W(F_{\Gamma(t)}^{***}) = W(F_{\Gamma(t)}^{***}) + \left[ 4c_{1}(b c_{1} + 2c_{2}) - 8b(e_{1} + e_{2}) \right] (\Gamma(t)) \alpha |\Delta t + o(|\Delta t|), \]  

(65)

Since \( \alpha \in \mathbb{C} \cong \mathbb{R}^2 \), our claim is proved.

It remains to prove \( \dim_{\mathbb{R}} \Psi(U \times E) = 4 \). Notice that \( \dim_{\mathbb{R}} \mathcal{K} = 4 \), \( \dim_{\mathbb{R}}(\mathcal{K}^{*}) \geq 2 \), and that the map defined by \( (\mathcal{K}^{*} - \mathcal{E}) \times \partial \mathbb{H}^2 \rightarrow \mathcal{K}, (F, p) \mapsto F_{p}^{***} \) is (Nash) algebraic. Then it suffices to show that this map is injective, i.e., for any two distinct points \( (c_{1}, c_{3}, e_{1}, e_{2}), (\tilde{c}_{1}, \tilde{c}_{3}, \tilde{e}_{1}, \tilde{e}_{2}) \in \mathcal{K}^{*} \), which are sufficiently close to \( (c_{1}^{(0)}, c_{3}^{(0)}, e_{1}^{(0)}, e_{2}^{(0)}) \), and for any two points \( p, \tilde{p} \in \partial \mathbb{H}^2 \), which are sufficiently close to 0 in \( \partial \mathbb{H}^2 \),

\[ (F_{c_{1}, c_{3}, e_{1}, e_{2}})_{p}^{***} \neq (F_{\tilde{c}_{1}, \tilde{c}_{3}, \tilde{e}_{1}, \tilde{e}_{2}})_{\tilde{p}}^{***}. \]  

(66)

If this can be proved, it follows \( \dim_{\mathbb{R}} \Psi(U \times E) = 4 \).

Recall that for a fixed \( F \), we write

\[ F_{p}^{***} = H_{p} \circ \tau_{p} \circ F \circ \sigma_{p} \circ G_{p}, \]  

(67)
where \( \sigma_p \in Aut(\mathbb{H}^2) \) and \( \tau_p \in Aut(\mathbb{H}^5) \) are defined in (18), \( G_p \in Aut_0(\mathbb{H}^2) \) and \( H_p \in Aut_0(\partial \mathbb{H}^3) \).

In case (66) does not hold, i.e., we have \((F_{c_1,c_3,e_1,e_2})_{p}^{***} = (F_{c_1,c_3,e_1,e_2})_{p_0}^{***} \). By (67), we write

\[
H_p \circ \tau_p \circ F_{c_1,c_3,e_1,e_2} \circ \sigma_p \circ G_p = \tilde{H}_p \circ \tilde{\tau}_p \circ F_{c_1,c_3,e_1,e_2} \circ \tilde{\sigma}_p \circ \tilde{G}_p,
\]

i.e.,

\[
F_{c_1,c_3,e_1,e_2} = \tau_p^{-1} \circ H_p^{-1} \circ \tilde{H}_p \circ \tilde{\tau}_p \circ F_{c_1,c_3,e_1,e_2} \circ \tilde{\sigma}_p \circ \tilde{G}_p \circ G_p^{-1} \circ \sigma_p^{-1} = (F_{c_1,c_3,e_1,e_2})_{p_0}^{***},
\]

where \( p_0 = \tilde{\sigma}_p \circ \tilde{G}_p \circ G_p^{-1} \circ \sigma_p^{-1}(0) \).

Notice from (67) that there is \( \delta > 0 \) such that as \( p \to 0 \), \( \sigma_p, G_p, \tau_p, H_p \) all converge to the identity maps in \( Aut(\mathbb{H}^2) \) and \( Aut(\mathbb{H}^5) \) respectively. We apply this fact to (68) to conclude that for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \((c_1, c_3, e_1, e_2), (\tilde{c}_1, \tilde{c}_3, \tilde{e}_1, \tilde{e}_2) \in \mathcal{K}^* \) with

\[
dist((c_1, c_3, e_1, e_2), (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})) < \delta, \quad \text{dist}((\tilde{c}_1, \tilde{c}_3, \tilde{e}_1, \tilde{e}_2), (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})) < \delta,
\]

we must have \( |p_0| < \epsilon \). We can choose \( \epsilon \) to be the same as in Lemma 4.1. By applying Lemma 4.1 to (68) to conclude \( F_{c_1,c_3,e_1,e_2} = F_{c_1,c_3,e_1,e_2} \). This contracts with the fact that \((c_1, c_3, e_1, e_2)\) and \((\tilde{c}_1, \tilde{c}_3, \tilde{e}_1, \tilde{e}_2)\) are distinct. Hence (66) is proved.

**Corollary 4.3** *(Local version of Theorem 1.1(ii))* For any \( P^{(0)} = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^* - \mathcal{E} \) where \( \mathcal{E} \) is defined in Lemma 3.3, there is a neighborhood \( U \) of \( P^{(0)} \) in \( \mathcal{K}^* - \mathcal{E} \) such that \( \forall (c'_1, c'_3, e'_1, e'_2), (c''_1, c''_3, e''_1, e''_2) \in U \) such that \( F_{c'_1,c'_3,e'_1,e'_2} \) and \( F_{c''_1,c''_3,e''_1,e''_2} \) are equivalent, we have \((c_1, c_3, e_1, e_2) = (c'_1, c'_3, e'_1, e'_2)\).

**Proof:** Let \( U_1 \) be a neighborhood of \( P^{(0)} \) in \( \mathcal{K}^* - \mathcal{E} \), \( E \) a neighborhood of \( 0 \) in \( \partial \mathbb{H}^2 \) and \( \mathcal{V} \) a neighborhood of \( P^{(0)} \) in \( \mathcal{K} \) as in Lemma 4.2. Let \( U \) be a neighborhood of \( P^{(0)} \) in \( \mathcal{K}^* - \mathcal{E} \) and \( c > 0 \) be a constant as in Lemma 4.1. We choose \( U_1, E = \{(z, u + i|z|^2) \in \partial \mathbb{H}^2 \mid |z| < c, |u| < c\}, V \) such that \( U_1 \subset U \) and \( V \cap (\mathcal{K}^* - \mathcal{E}) \subset U \). Then by Lemma 4.2, we have \( F_{c'_1,c'_3,e'_1,e'_2} = (F_{c''_1,c''_3,e''_1,e''_2})_{p}^{***} \) with \( |p| < c \), and by Lemma 4.1, \((c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) = (c'_1, c'_3, e'_1, e'_2)\).

\( \square \)

## 5 The proof of Theorem 1.1

Before proving Theorem 1.1, we mention a fact. Let \( \sigma_a \) and \( \sigma_b \in Aut(\partial \mathbb{H}^2) \) defined as in (18) and \( F \in Rat(\mathbb{H}^2, \mathbb{H}^5) \), then we can define a family of automorphism \( \Theta_s = \sigma_{sb+(1-s)a}, 0 \leq s \leq 1 \) with...
s ≤ 1, and Ψ_s = \tau^F_{sb+(1-s)a} ∈ Aut(\partial \mathbb{H}^5) defined as in (18) so that Ψ_s ∘ F ∘ Θ_s ∈ Rat(\mathbb{H}^2, \mathbb{H}^5) satisfies Θ_0 = \sigma_a, Θ_1 = \sigma_b and

\[ \Psi_s ∘ F ∘ Θ_s(0) = 0, \ \forall s \in [0, 1]. \] (69)

**Proof of Theorem 1.1:** For any \( F ∈ Rat(\mathbb{H}^2, \mathbb{H}^5) \) with degree 2, by [CJX06] and Lemma 3.3, \( F \) is equivalent to another map \( F_{c, e_3, e_1, e_2} ∈ K^* \) with the minimum property (9). By Lemma 3.2 and 3.4, Theorem 1.1(i) is proved.

It remains to prove Theorem 1.1(ii). We need to show: if two distinct maps \( F_{c, e_3, e_1, e_2} ∈ K^* \) and \( \tilde{F}_{\tilde{c}, \tilde{e}_3, \tilde{e}_1, \tilde{e}_2} ∈ K^* \) are equivalent, then

\[ (\tilde{c}_1, \tilde{c}_3, \tilde{c}_1, \tilde{c}_2) = (c_1, c_3, c_1, c_2). \] (70)

We assume that \( (c_0, c_3, e_1, e_2) \notin E \) where \( E \) is defined in Lemma 3.3; otherwise these two maps \( F_{c, e_3, e_1, e_2} \) and \( \tilde{F}_{\tilde{c}, \tilde{e}_3, \tilde{e}_1, \tilde{e}_2} \) cannot be equivalent.

**Step 1. Construct a curve \( \hat{L}_0 \)** Since \( F_{c, e_3, e_1, e_2} \) and \( \tilde{F}_{\tilde{c}, \tilde{e}_3, \tilde{e}_1, \tilde{e}_2} \) are equivalent,

\[ F_{c, e_3, e_1, e_2} = \Psi \circ F_{c, e_3, e_1, e_2} \circ Θ \] (71)

where \( Θ ∈ Aut(\mathbb{H}^2) \) and \( Ψ ∈ Aut(\mathbb{H}^5) \). Notice \( Ψ ∘ F_{c, e_3, e_1, e_2} ∘ Θ(0) = 0 \) holds.

We take a real analytic curve \( L = L(s) ∈ K^* - E \), \( 0 ≤ s < 1 \), such that \( L(0) = (c_0, c_3, e_1, e_2) \). In fact, since \( (c_0, c_3, e_1, e_2) \notin E \) and \( E \) is closed, \( L \) could be taken in a neighborhood of \( (c_0, c_3, e_1, e_2) \).

By using automorphisms of balls, Cayley transformations and (69), we can take a real analytic family of automorphisms \( Θ_s ∈ Aut(\partial \mathbb{H}^2) \), \( Ψ_s ∈ Aut(\partial \mathbb{H}^5) \), \( s ∈ [0, 1] \), such that when \( s = 0, Θ_0 = Θ, Ψ_0 = Ψ \); when \( s ∈ (0, 1) \), \( Θ_s(0) ≠ ∞, Ψ_s ∘ F_{L(s)} ∘ Θ_s(0) = 0 \); when \( s = 1, Θ_1 = Id, Ψ_1 = Id \). Then we define

\[ \hat{L}_0(s) := Ψ_s ∘ F_{L(s)} ∘ Θ_s ∈ Rat(\mathbb{H}^2, \mathbb{H}^5), \ \ 0 ≤ s ≤ 1, \]

such that \( \hat{L}_0(s)(0) = 0 \) for all \( s \), \( F_{L(0)} = Ψ ∘ F_{L(0)} ∘ Θ \) and \( \hat{L}_0(1) = L(1) \). Our goal is to show: \( \hat{L}_0(s) = L(s), \ \forall s ∈ [0, 1] \), so that \( \hat{L}_0(0) = L(0) \), i.e., (70), holds.

**Step 2. Define a curve \( \hat{L}(s) \)** Notice that \( \hat{L}_0 \) must be in \( K \), namely, \( F_{\hat{L}_0(s)} \) may geometric rank one at the origin for all \( s ∈ [0, 1] \), so that \( (F_{\hat{L}_0(s)})^{∗∗∗} \) is well defined for all \( s ∈ [0, 1] \).
Recall Θ_s(0) ≠ ∞ for any s ∈ (0, 1] and Θ_1 = Id. Then for any s ∈ (0, 1], we denote ψ(s) := Θ_s(0) ∈ ∂H^2 with ψ(1) = 0, so that Θ_s = σ_{ψ(s)} ∘ G_s where σ_{ψ(s)} is defined as in (18) and G_s ∈ Auto_0(∂H^2), i.e., we have a continuous map ψ(s) ∈ ∂H^2 such that ψ(1) = 0 and

\[ (F_{L_0(s)})^{***} = (F_L(s))^{***}_{ψ(s)}, \quad ∀s ∈ (0, 1], \quad \text{and} \quad (F_{L_0(1)})^{***} = F_L(1). \quad (72) \]

Even though \((F_{L_0(s)})^{***}\) is in \(K\) for any \(s ∈ (0, 1]\), it may not be in \(K^*\) because the minimum property (9) may not be satisfied. We claim that \((F_{L_0(s)})^{***}\) is equivalent to another map \(F_{L(s)} ∈ K^*\). More precisely, we want to find \(q(s) ∈ ∂H^2\) so that

\[ F_{L(s)} := (F_{L_0(s)})^{***}_{q(s)} ∈ K^*, \quad ∀s ∈ (0, 1]. \quad (73) \]

To define such \(q(s)\), we consider several cases below.

If \(s = 1\), since \(F_{L_1(1)} ∈ K^*\) and \(ψ(1) = 0\), we define \(q(1) = 0\).

If \(s ∈ (0, 1]\) at which the minimum property (9) holds, we define \(q(s) = 0\).

If \(s ∈ (0, 1]\) at which (9) does not hold, we consider a continuous curve \(Γ^{(s)}(t) ∈ ∂H^2 − Ξ_F, \quad 0 ≤ t ≤ 1\), with \(Γ^{(s)}(0) = 0\) such that the function value \(W((F_{L_0(s)})^{***}_{Γ^{(s)}(t)})\) is increasing along \(Γ^{(s)}\). We denote by \(ℓ_s\) the infimum of \(W((F_{L_0(s)})^{***}_{Γ^{(s)}})\) over all such curves. Then there exists a sequence of curves \(Γ_{m}^{(s)}\) in \(∂H^2\) such that

\[ ℓ_s = \lim_{m → ∞} W((F_{L(s)})^{***}_{Γ_{m}^{(s)}(1)}). \quad (74) \]

Since \(W((F_{L_0(s)})^{***}_{p}) = c_1(p)^2 − e_1(p) − e_2(p)\), the decreasing property implies \(c_1(p), −e_1(p)\) and \(−e_2(p)\) are bounded (cf. [CJX06, Step 1, §4]), so that \((F_{L_0(s)})^{***}_{Γ_{m}^{(s)}(1)}\), regarded as a point, is inside \(K\) and is contained a compact subset of \(K\) that is independent of \(Γ_{m}^{(s)}\). Therefore, by taking subsequences, we may assume that the limit \(\lim_{m → ∞}(F_{L_0(s)})^{***}_{Γ_{m}^{(s)}(1)}\) exists as a point in \(K^*\) and that \(\lim_{m → ∞} Γ_{m}^{(s)}(1) ∈ ∂H^2\) exists. Let us define

\[ F_{L(s)} := \lim_{m → ∞} (F_{L_0(s)})^{***}_{Γ_{m}^{(s)}(1)} ∈ K^*. \quad (75) \]

It remains to show that \(q(s) ∈ ∂H^2\) can be defined such that \(F_{L(s)} = (F_{L_0(s)})^{***}_{q(s)}\).

By the choice of \(L(1)\) and Corollary 4.3, there exists a neighborhood \(U\) of \(L(1)\) in \(K^*\), such that if a point \((c_1, c_3, e_1, e_2) ∈ U\) such that \(F_{c_1,c_3,e_1,e_2}\) and \(F_{L(1)}\) are equivalent, then \((c_1, c_3, e_1, e_2) = L(1)\).

Let us consider \(K^* ∩ \overline{B}^4(\hat{L}_0(s), r)\), the intersection of \(K^*\) with the sphere in \(\mathbb{C}^4\) which is centered at \(\hat{L}_0(s)\) with radius \(r\). We also consider \(K^* ∩ \overline{B}^2(\hat{L}_0(s), r)\), the intersection of \(K^*\)
with the sphere in $\mathbb{C}^2$ which is centered at $\hat{L}_0(s)$ with radius $r$. We take $r$ so small that $K^* \cap B^2(\hat{L}_0(s), r) \subset U$.

**Step 3. Claim on $F_{\hat{L}(s)} \rightarrow F_{L_0(s)}$** Regarding $F_{\hat{L}(s)}$ as points in $\mathcal{K}$, we claim:

$$\text{dist}\left(F_{\hat{L}(s)}, F_{L_0(s)}\right) \rightarrow 0, \text{ as } s \rightarrow 1.$$ (76)

Suppose (76) is not true. Then there exists a sequence $s_k \rightarrow 1$ such that

$$\text{dist}\left(F_{\hat{L}(s_k)}, F_{L_0(s_k)}\right) \geq \delta_0, \text{ as } k \rightarrow \infty.$$ (77)

for a certain $\delta_0 > 0$. By (75), we can take integer $m_{s_k}$ for each $s_k$ such that

$$0 \leq W((F_{L_0(s_k)})_{m_{s_k}}^{(s_k)}(L(1)), \ell_{s_k} < \frac{1}{k}, \text{ and dist}\left((F_{L_0(s_k)})_{m_{s_k}}^{(s_k)}(L(1)), F_{\hat{L}(s_k)}\right) < \frac{1}{k}.$$ (78)

By (77) we have

$$\text{dist}\left((F_{L_0(s_k)})_{m_{s_k}}^{(s_k)}(L(1)), F_{\hat{L}(s_k)}\right) \geq \frac{\delta_0}{2}.$$ (79)

Then we can choose $r < \frac{\delta_0}{2}$. Then $\{(F_{L_0(s_k)})_{m_{s_k}}^{(s_k)}(L(1))\}_{t \in [0, 1]}$, regarded as a curve in $\mathcal{K}$ initiated from the point $F_{L_0(s_k)}$, must be across through the sphere $(\mathcal{K} \cap \partial B^4(\hat{L}_0(s_k), r))$, i.e.,

$$\{(F_{L_0(s_k)})_{m_{s_k}}^{(s_k)}(L(1)) \cap (\mathcal{K} \cap \partial B^4(\hat{L}_0(s_k), r)) \neq \emptyset.$$ (80)

Let $Q^{(s_k)}$ be a point in the intersection (80) and then $Q^{(s_k)} = (F_{L_0(s_k)})_{m_{s_k}}^{(s_k)}(L(1))$ for some $t_k \in [0, 1]$. By taking subsequences, we assume $Q := \lim_{k \rightarrow \infty} Q^{(s_k)}$ exists. By the construction, we see that the $F_Q$ is equivalent to $F_{L(1)}$ and

$$Q \in \mathcal{K}^*, \text{ and dist}(Q, L(1)) = r.$$ Since $Q \in \mathcal{K}^* \cap \partial B^2(\hat{L}_0(1), r) \subset U$, by Corollary 4.3, $Q = L(1)$, i.e., $\text{dist}(Q, L(1)) = 0$, but this is a contradiction. Claim (76) is proved.

**Step 4. Proof of $\hat{L}(s) \equiv L(s)$** From (76), we have

$$\text{dist}\left(F_{\hat{L}(s)}, F_{L(s)}\right) \rightarrow 0, \text{ as } s \rightarrow 1.$$
Since both $F_{L(s)} \in K^*$ and $F_{L(s)} \in K^* - \mathcal{E}$ where $s \in (s_0, 1]$ for some $s_0 > 0$ such that $0 \leq 1 - s_0$ is sufficiently small, by Corollary 4.3 and the choice of $L(1)$, we conclude

$$F_{L(s)} = F_{L(s)}, \quad \forall s \in (s_0, 1].$$

Repeating this process. Finally by continuity $F_{L(s)} = F_{L(s)}, \forall s \in [0, 1]$. When restricted at 0, $F_{L(0)} = F_{L(0)} = F_{L(0)}$, so that (70) is proved. □

**Acknowledgments**  This work was started when the first author was visiting the School of Mathematical Sciences, Wuhan University, China, in the summer of 2004. The authors are indebted to Professor Hua Chen for his support and arrangement, which made the visit possible.

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